Chiral four-body interactions in nuclear matter

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- Hierarchy of nuclear forces in chiral effective field theory
- Leading order chiral 4N-forces in nuclear and neutron matter
- $\Delta(1232)$-excitation of two nucleons: 3-ring, 2-ring and 1-ring diagrams
- Twofold $\Delta(1232)$-excitation of one nucleon: 2-ring and 1-ring diagrams
- Exact calculation of 3-body contact-interaction to second order

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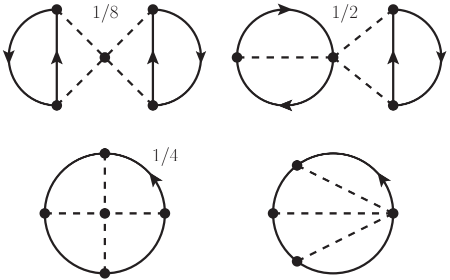
- Chiral EFT: nuclear forces are organized hierarchically
- For observables: 2-body force $>>$ 3-body force $>>$ 4-body force ...
- 2-body interaction: universal NN-potential $V_{\rm low-k}$, chiral N^{3}LO potential
- 3-body interaction: leading order chiral 3N-force, contact + 1π + 2π-exch.
- Low-momentum 2N and 3N interactions: good results for nuclear and neutron matter in many-body perturbation theory ($\rightarrow$ second order)
- Recent work by Darmstadt group (A. Schwenk et al.): sizeable attraction from subleading 3N-interaction in neutron matter, -10 MeV at ρ_n=0.2fm^{-3}
- Chiral 4N-interaction constructed by Epelbaum via method of unitary transformations: project πN-dynamics into purely nucleonic subspace
- Here: exploratory study of long-range 4N-interaction mediated by π-exchange in nuclear/neutron matter, include virtual Δ(1232)-isobars

Leading order terms related to 4π-vertex

- Method of unitary transformations (Epelbaum) gives "induced" 4N-forces from reducible diagrams, consider first "genuine" 4-body interactions
- At leading order: determined by chiral 4π-vertex and $NN3\pi$-vertex



- 2-ring (Hartree) diagrams: integrals over four Fermi spheres factorize

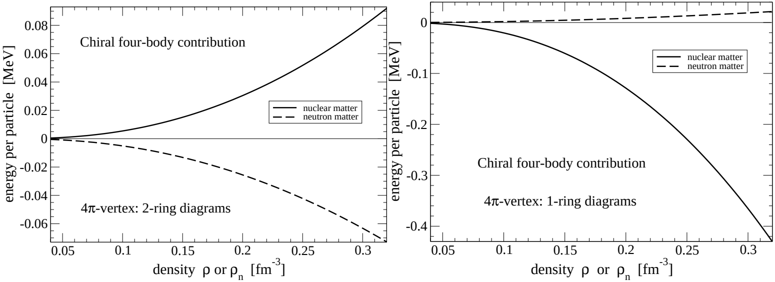
$$\bar{E}(\rho) = \frac{9g_A^4 m_\pi^7 u}{(4\pi f_\pi)^6}\left[u^2 - \frac{1}{2} - 2u \arctan 2u + \left(1 + \frac{1}{8u^2}\right)\ln(1+4u^2)\right]^2,$$

$$\bar{E}_n(\rho_n) = -\frac{3g_A^4 m_\pi^7 u}{2(4\pi f_\pi)^6}\left[u^2 - \frac{1}{2} - 2u \arctan 2u + \left(1 + \frac{1}{8u^2}\right)\ln(1+4u^2)\right]^2$$

$u = k_f/m_\pi$ for nuclear matter $\rho = 2k_f^3/3\pi^2$, $u = k_n/m_\pi$ for neutron matter $\rho_n = k_n^3/3\pi^2$

Leading order terms related to 4π-vertex

- Adding diagrams: only a constant $-3m_\pi^2/f_\pi^2$ remains from $\pi\pi$-interaction



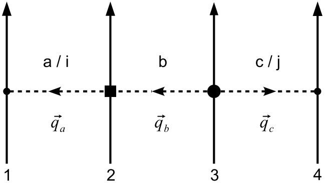
- 1-ring Fock diagrams: larger than Hartree contributions (unusual feature)

$$\bar{E}(\rho) = \frac{3g_A^4}{16 f_\pi^6 \rho} \int\limits_{|\vec{p}_j|<k_f} \frac{d^{12}p}{(2\pi)^{12}} \frac{1}{m_\pi^2 + \vec{q}_1^{\,2}} \frac{1}{m_\pi^2 + \vec{q}_2^{\,2}} \frac{1}{m_\pi^2 + \vec{q}_3^{\,2}} \frac{1}{m_\pi^2 + (\vec{q}_1 + \vec{q}_2 + \vec{q}_3)^2}$$
$$\times \Big\{ -8[\vec{q}_1\cdot(\vec{q}_2\times\vec{q}_3)]^2 + m_\pi^2 \Big[-4\vec{q}_1^{\,2}\,\vec{q}_3^{\,2} - 4\vec{q}_1^{\,2}(\vec{q}_2^{\,2} + \vec{q}_1\cdot\vec{q}_3)$$
$$+2\vec{q}_1^{\,2}\,\vec{q}_2\cdot(2\vec{q}_1 - 5\vec{q}_3) + 8(\vec{q}_1\cdot\vec{q}_2)^2 + \vec{q}_1\cdot\vec{q}_3\,\vec{q}_2\cdot(8\vec{q}_1 + 3\vec{q}_2) - 6\vec{q}_1\cdot\vec{q}_2\,\vec{q}_2\cdot\vec{q}_3 \Big] \Big\}$$

In contrast to this: $\bar{E}_n(\rho_n) \sim m_\pi^2$ vanishes in chiral limit

- With less than 0.4 MeV for $\rho \leq 0.32\,\text{fm}^{-3}$ this 4N-interaction is negligible

- Method of unitary transformations generates "reducible" 4N-forces: pion-exchanges in combination with a short-range contact-coupling C_T



- Represent 4N-interaction as a product of 4 vertices and 3 propagators

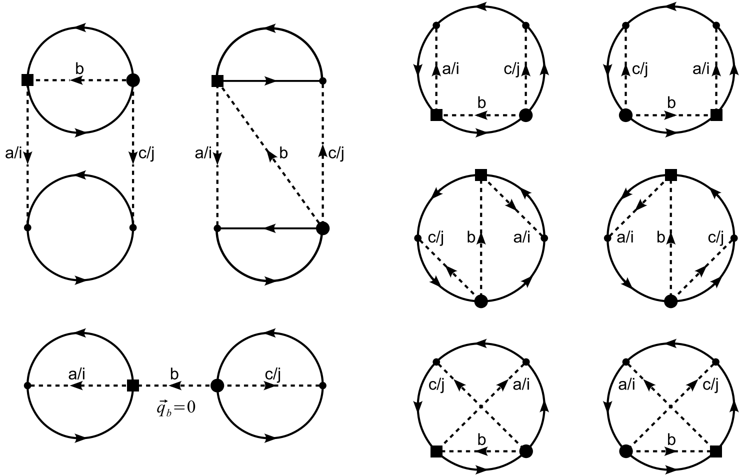
$$V^n = -\frac{C_T^2 g_A^2}{f_\pi^2}\,\sigma_1^i\,(\vec\sigma_2\times\vec q_b)^i\,\tau_2^b\,\frac{1}{(m_\pi^2+\vec q_b^{\,2})^2}\,(\vec\sigma_3\times\vec q_b)^j\,\tau_3^b\,\sigma_4^j\,, \qquad V^{l,k} = \frac{C_T g_A^{2,4}}{8f_\pi^4}\,\vec\sigma_1\cdot\vec q_a\,\tau_1^a\,\ldots$$

$$V^b = -\frac{g_A^4}{32 f_\pi^6}\,\vec\sigma_1\cdot\vec q_a\,\tau_1^a\,\frac{1}{m_\pi^2+\vec q_a^{\,2}}\,\epsilon^{abd}\,\tau_2^d\,\frac{1}{m_\pi^2+\vec q_b^{\,2}}$$
$$\times\left[\epsilon^{bce}\,\tau_3^e\,\vec q_b\cdot\vec q_c + \delta^{bc}\,\vec\sigma_3\cdot(\vec q_b\times\vec q_c)\right]\frac{1}{m_\pi^2+\vec q_c^{\,2}}\,\vec\sigma_4\cdot\vec q_c\,\tau_4^c\,,$$

$$V^a = \frac{g_A^6}{16 f_\pi^6}\,\vec\sigma_1\cdot\vec q_a\,\tau_1^a\,\frac{1}{m_\pi^2+\vec q_a^{\,2}}\left[\epsilon^{abd}\,\tau_2^d\,\vec q_a\cdot\vec q_b + \delta^{ab}\,\vec\sigma_2\cdot(\vec q_a\times\vec q_b)\right]$$
$$\times\frac{1}{(m_\pi^2+\vec q_b^{\,2})^2}\left[\epsilon^{bce}\,\tau_3^e\,\vec q_b\cdot\vec q_c + \delta^{bc}\,\vec\sigma_3\cdot(\vec q_b\times\vec q_c)\right]\frac{1}{m_\pi^2+\vec q_c^{\,2}}\,\vec\sigma_4\cdot\vec q_c\,\tau_4^c\,,$$

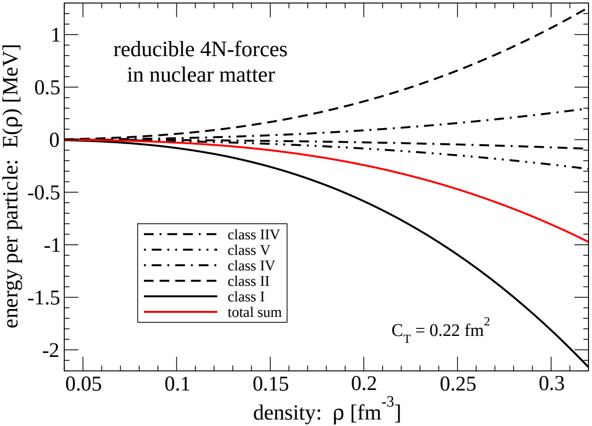
Reducible chiral 4N-interactions

- Zero if N-line is closed to itself: 4-ring and 3-ring diagrams vanish
- Evaluate 2-ring and 1-ring diagrams for 5 classes of 4N-interactions



- Most integrals over 4 Fermi spheres can be reduced to double-integrals

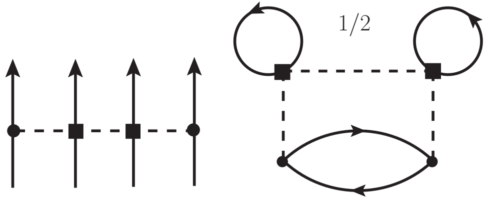
- Numerical results for energy per particle



- Cancelations between individual classes of contributions: net effect of reducible chiral 4N-forces in nuclear/neutron matter less than 1 MeV
- Only class I (2-ring) contributes to neutron matter: $\bar{E}_n(2\rho_0) = -1.18\,\mathrm{MeV}$

Inclusion of virtual $\Delta(1232)$-isobars

- Phenomenology: virtual $\Delta(1232)$-excitation of two N's and π-exchanges



- Combine direct + crossed $\Delta(1232)$-excitation into a 2π-contact vertex

$$\frac{ig_A^2}{f_\pi^2 \Delta}\left[\delta_{ab}\,\vec{q}_a \cdot \vec{q}_b - \frac{1}{4}\epsilon_{abc}\tau_c\,\vec{\sigma}\cdot(\vec{q}_a \times \vec{q}_b)\right]$$

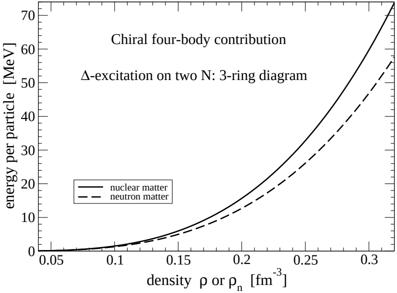
with mass-splitting $\Delta = 293\,\text{MeV}$ and coupling const. ratio $g_{\pi N\Delta}/g_{\pi NN} = 3/\sqrt{2}$

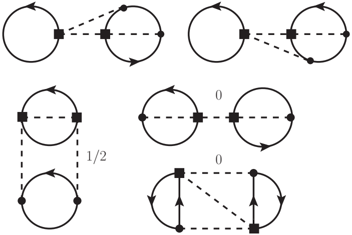
- Also a leading order 4N-force: if Δ is small scale comparable to k_f, m_π
- 3-ring diagram is easily calculated

$$\bar{E}(\rho) = \frac{g_A^6 m_\pi^9 u^3}{(2\pi f_\pi)^6 \Delta^2}\left\{\frac{16u^6}{9} - 12u^4 + \frac{20u^2}{3} + \frac{70u^3}{3}\arctan 2u - \left(12u^2 + \frac{5}{3}\right)\ln(1+4u^2)\right\},$$

$$\bar{E}_n(\rho_n) = \frac{g_A^6 m_\pi^9 u^3}{(2\pi f_\pi)^6 \Delta^2}\left\{\frac{4u^6}{27} - u^4 + \frac{5u^2}{9} + \frac{35u^3}{18}\arctan 2u - \left(u^2 + \frac{5}{36}\right)\ln(1+4u^2)\right\}$$

- Sizeable repulsion from these 4N-processes: reduction by "Fock" terms?





- 2-ring diagrams of upper set: integrals factorize by use of tensors

$$\bar{E}(\rho) = -\frac{g_A^6 m_\pi^9}{4(2\pi f_\pi)^6 \Delta^2} \int_0^u dx \left[2G_S(x)H_S(x) + G_T(x)H_T(x) \right],$$

$$\bar{E}_n(\rho_n) = -\frac{g_A^6 m_\pi^9}{24(2\pi f_\pi)^6 \Delta^2} \int_0^u dx \left[G_S(x)H_S(x) + 2G_T(x)H_T(x) \right]$$

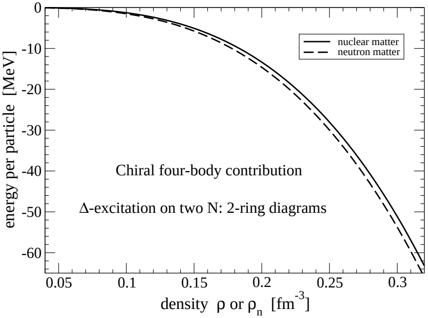
$G_{S,T}(x), H_{S,T}(x)$ are auxiliary functions involving $\arctan(u \pm x)$ and $\ln \frac{1+(u+x)^2}{1+(u-x)^2}$ which arise from Fermi-sphere integral over π-propagators

Inclusion of virtual $\Delta(1232)$-isobars

- Other 2-ring diagram: integrate over shifted Fermi spheres

$$\bar{E}(\rho) = \frac{g_A^6 m_\pi^9}{(2\pi f_\pi)^6 \Delta^2 u^3} \int_0^u dx \int_0^u dy \, \frac{x^3(u^2-y^2)}{(1+4x^2)^2} (u-x)^2 (2u+x) \Big\{ 7xy(1+4y^2)^2$$
$$+16x^3 \Big[10y - \frac{136y^3}{3} - 7x^2 y - 5\arctan(2x+2y) + 5\arctan(2x-2y) \Big]$$
$$+\frac{1}{16}(4x^2-4y^2-1)\Big[7(1+4y^2)^2 + 8x^2(14x^2-28y^2-13)\Big] \ln\frac{1+4(x+y)^2}{1+4(x-y)^2} \Big\}$$

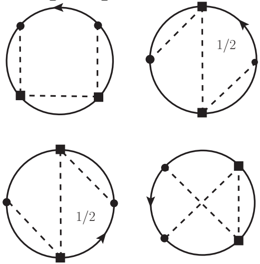
$2m_\pi x$, $2m_\pi y$ are momentum transfers carried by pions, $\int_0^u dy \ldots$ is still solvable

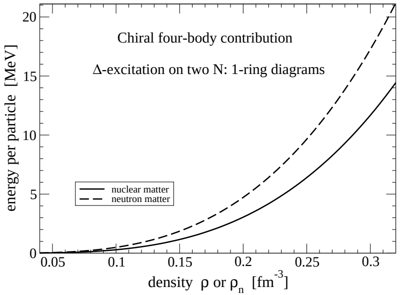


- Attractive 2-ring diagrams nearly balance strong repulsion from 3-ring

Inclusion of virtual Δ(1232)-isobars

- 1-ring diagrams remain:





$$\bar{E}(\rho)^U = \frac{g_A^6 m_\pi^9}{8(4\pi f_\pi)^6 \Delta^2 u^3} \int_0^u dx\, \frac{1}{x}\Big[8G_S^3(x) + 9G_S(x)G_T^2(x) + G_T^3(x)\Big]\,,$$

$$\bar{E}(\rho)^Z = \frac{g_A^6 m_\pi^9}{(4\pi f_\pi)^6 \Delta^2 u^3} \int_0^u dx \int_0^u dy\, \Big\{2G_S(x)G_S(y)\Big[4xy - L\Big] + G_S(x)G_T(y)$$

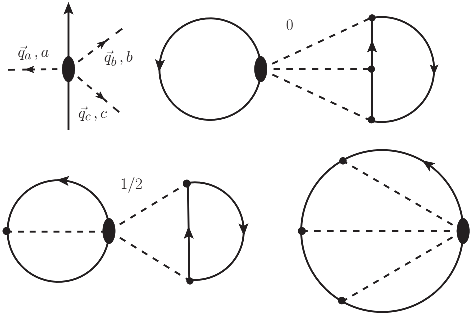
$$\times \Big[5xy - \frac{3x}{y}(1+x^2) + \frac{L}{4}\Big(\frac{3}{y^2}(1+x^2)^2 - 2 - 6x^2 + 3y^2\Big)\Big] + G_T(x)G_T(y)$$

$$\times \Big[\frac{xy}{8} + \frac{3}{8y}\Big(\frac{3}{2x} + 2x - x^3\Big) + \frac{L}{32}\Big(\frac{3}{y^2}\Big(x^4 - x^2 - 5 - \frac{3}{2x^2}\Big) - 1 - 3x^2\Big)\Big]\Big\}\,,$$

$$\bar{E}(\rho)^X = \frac{3g_A^6}{f_\pi^6 \Delta^2 \rho} \int_{|\vec{p}_j|<k_f} \frac{d^{12}p}{(2\pi)^{12}} \frac{\vec{q}_1^{\,2}\,(\vec{q}_2\cdot\vec{q}_3)^2}{(m_\pi^2+\vec{q}_1^{\,2})(m_\pi^2+\vec{q}_2^{\,2})(m_\pi^2+\vec{q}_3^{\,2})}\,, \qquad L = \ln\frac{1+(x+y)^2}{1+(x-y)^2}$$

Twofold $\Delta(1232)$-excitation on one nucleon

- Take into account direct coupling of pion to $\Delta(1232)$-isobar: $\Delta\Delta\pi$-vertex



- Condense $\Delta\Delta\pi$-dynamics into a totally symmetrized 3π-contact vertex

$$\frac{g_A^3}{40f_\pi^3\Delta^2}\Big\{-75\epsilon_{abc}\,\vec{q}_a\cdot(\vec{q}_b\times\vec{q}_c)+\vec{q}_a\cdot\vec{q}_b\,\vec{\sigma}\cdot\vec{q}_c\,(18\delta_{ab}\tau_c-7\delta_{ac}\tau_b-7\delta_{bc}\tau_a)$$
$$+\vec{q}_a\cdot\vec{q}_c\,\vec{\sigma}\cdot\vec{q}_b\,(18\delta_{ac}\tau_b-7\delta_{ab}\tau_c-7\delta_{bc}\tau_a)+\vec{q}_b\cdot\vec{q}_c\,\vec{\sigma}\cdot\vec{q}_a\,(18\delta_{bc}\tau_a-7\delta_{ac}\tau_b-7\delta_{ab}\tau_c)\Big\}$$

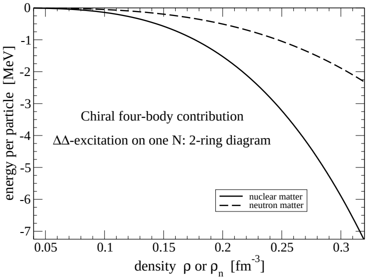
using relation $T_a\Theta_b T_c^\dagger=\frac{1}{3}(5i\,\epsilon_{abc}-\delta_{ab}\tau_c+4\delta_{ac}\tau_b-\delta_{bc}\tau_a)$ with $\vec{\Theta}$ the Δ-isospin operator and coupling ratio $g_{\pi\Delta\Delta}/g_{\pi NN}=1/5$ of quark model

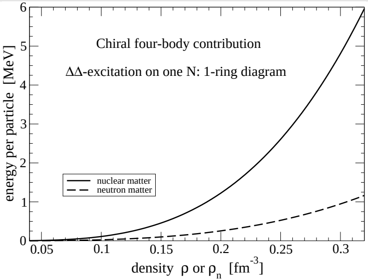
Twofold $\Delta(1232)$-excitation on one nucleon

- Corresponding 2-ring and 1-ring diagrams almost cancel each other

$$\bar{E}(\rho)^{2r} = \frac{5g_A^6 m_\pi^9 u}{3(4\pi f_\pi)^6 \Delta^2}\left[\frac{4u^4}{3} - 6u^2 + 2 + 10u \arctan 2u - \left(\frac{9}{2} + \frac{1}{2u^2}\right)\ln(1+4u^2)\right]$$
$$\times\left[12u^2 - 2 - \frac{16u^4}{3} - 16u \arctan 2u + \left(6 + \frac{1}{2u^2}\right)\ln(1+4u^2)\right],$$

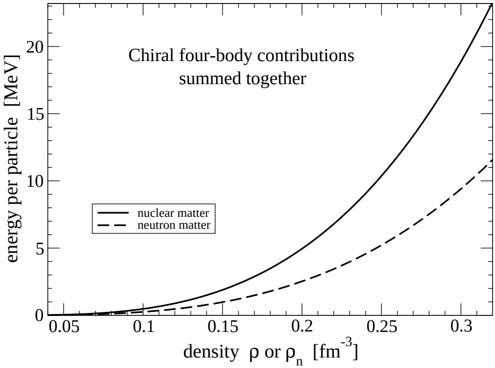
$$\bar{E}(\rho)^{1r} = \frac{3g_A^6 m_\pi^9}{(4\pi f_\pi)^6 \Delta^2 u^3}\int_0^u dx \int_0^u dy \left\{\frac{5}{4}G_S(x)G_S(y)\left[4xy - L\right] + \frac{5}{8}G_S(x)G_T(y)\right.$$
$$\times\left[\frac{3x}{y}(1+x^2) - 5xy + \frac{L}{4}\left(2 + 6x^2 - 3y^2 - \frac{3}{y^2}(1+x^2)^2\right)\right]$$
$$\left.+\frac{3}{8}G_T(x)G_T(y)\left[xy - \frac{3x}{y}(1+x^2) + \frac{L}{4}\left(\frac{3}{y^2}(1+x^2)^2 + 2 - 3x^2\right)\right]\right\}$$





Total chiral four-body contribution

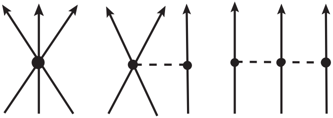
- Total result for chiral four-body contribution in nuclear and neutron matter



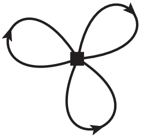
- At saturation density $\rho_0 = 0.16\,\text{fm}^{-3}$ moderate amount of $\bar{E} = 2.35\,\text{MeV}$, but increases strongly with density, approximately as ρ^3
- In neutron matter repulsive four-body contrib. reduced by about factor 2
- $\pi N\Delta$-system is strongly coupled with small mass-splitting $\Delta = 293\,\text{MeV}$
- Schwenk et al. find sizeable attraction from subleading chiral 3N-force
- One has to find a new balance between these unexpectedly large terms

Three-body contact interaction to second order

- Low-momentum NN-interactions ($p \leq 400$ MeV, without hard core) show good convergence properties in many-body perturbation theory
- 3N-force is essential to achieve saturation of nuclear matter

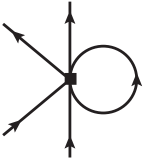


- Beyond leading order so far approximate treatment via effective 2N-force
- Exact calculation of simple 3-body contact interaction to second order



1st order contrib. to energy per particle: $C_3 = c_E / f_\pi^4 \Lambda_\chi$

$$\bar{E}(k_f) = -C_3 \frac{3\pi^2}{2k_f^3} \left(\frac{k_f^3}{6\pi^2} \right)^3 \cdot 12 = -\frac{C_3 k_f^6}{12\pi^4} = \Gamma_3 k_f^6$$

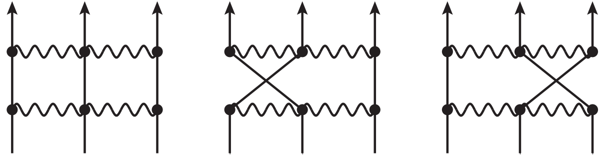


Density-dependent two-body contact coupling:

$$\delta C_0(\rho) = C_3 \left(\frac{k_f^3}{6\pi^2} \right) \cdot 6 = \frac{C_3 k_f^3}{\pi^2}$$

Three-body contact interaction to second order

- Three-particle rescattering in-medium, external momenta $|\vec{p}_j| < k_f$



$$B = \int \frac{d^3l_1 d^3l_2}{(2\pi)^6} \frac{3C_3 M}{\vec{l}_1^{\,2} + 3\vec{l}_2^{\,2}/4 - H/6 - i\epsilon} \Big[1 - \theta(k_f - |\vec{p} + \vec{l}_1 - \vec{l}_2/2|)\Big] \times \Big[1 - \theta(k_f - |\vec{p} + \vec{l}_2|)\Big] \Big[1 - \theta(k_f - |\vec{p} - \vec{l}_1 - \vec{l}_2/2|)\Big]$$

- Assign intermediate momenta: $\vec{p} + \vec{l}_1 - \vec{l}_2/2, \quad \vec{p} + \vec{l}_2, \quad \vec{p} - \vec{l}_1 - \vec{l}_2/2$ with $\vec{p} = (\vec{p}_1 + \vec{p}_2 + \vec{p}_3)/3$, set $|\vec{p}\,|/k_f = s < 1$
- $H = (\vec{p}_1 - \vec{p}_2)^2 + (\vec{p}_1 - \vec{p}_3)^2 + (\vec{p}_2 - \vec{p}_3)^2 < 9k_f^2$ is a Galilei invariant
- B is real, Pauli-blocking and energy conservation forbids imaginary part
- Vacuum term with no θ, 3 equal terms with one θ, 3 equal terms with $\theta\theta$
- Fermi-sphere integral of $\theta\theta\theta$-term is zero due to odd energy denominator

Three-body contact interaction to second order

- Vacuum loop using dim. regularization, $3 \to d$, prefactor λ^{6-2d}: $\Gamma(1-d)$

$$B_0 = \frac{\sqrt{3}C_3 M H^2}{(12\pi)^3}\left\{\left[\frac{1}{3-d} - \gamma_E + \ln 4\pi\right] + \frac{3}{2}(1+\ln 3) - \ln\frac{-H}{\lambda^2}\right\},$$

$$\mathrm{Re}\, B_0^{(ren)} = \frac{3\sqrt{3}C_3 M k_f^4}{(4\pi)^3} h^2 \left\{\frac{1}{2}(3-\ln 3) - 2\ln\frac{k_f}{\lambda} - \ln h + ct(\lambda)\right\}$$

- Need $O(p^4)$ counterterm prop. to H^2, set $H = 9k_f^2 h$, constraint $s^2 + h < 1$
- θ term: Integrate $\mathrm{Re}\sqrt{9\vec{l}_2^{\,2} - 2H}$ over shifted Fermi sphere $|\vec{p} + \vec{l}_2| < k_f$

$$\mathrm{Re}\, B_1 = \frac{3\sqrt{3}C_3 M k_f^4}{(4\pi)^3}\left\{\theta(\dots)\left[\frac{\sqrt{(1+s)^2 - 2h}}{10s}\left[16h^2 + h(9s^2 - 7s - 16)\right.\right.\right.$$
$$\left.\left.\left. + (1+s)^3(4-s)\right] - 3h^2 \ln\frac{1+s+\sqrt{(1+s)^2-2h}}{\sqrt{2h}}\right] + (s \to -s)\right\}$$

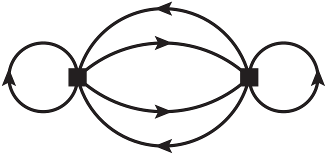
- $\theta\theta$ term $\mathrm{Re}\, B_2$: Integrate hole-hole “bubble” funct. (with logs and arctan) over Fermi sphere $|\vec{l}_2 - 2\vec{p}| < 2k_f$, lengthy analytical expression in (s, h)

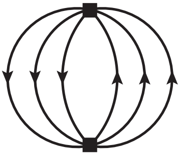
- Real part of $\theta\theta$ term with its dependence on the variables s and h

$$\begin{aligned}
\mathrm{Re}\, B_2 = \frac{9C_3 M k_f^4}{(4\pi)^4} \Bigg\{ & \frac{28}{5}(3-s^2) - \frac{22h}{5} + \bigg[h(7-15s^2) - 3h^2 - 5 + 10s^2 - \frac{33s^4}{5}\bigg] \\
& \times \ln\left|2s^2 - h + \frac{2}{3}\right| + \bigg[h^2\left(3 + \frac{1}{2s}\right) + h\left(15s^2 + 7s - 7 - \frac{5}{3s}\right) \\
& + \frac{33s^4}{5} + 2s^3 - 10s^2 - \frac{8s}{3} + 5 + \frac{86}{45s}\bigg] \ln|2(1+s)^2 - h| \\
& + \frac{16}{45s}(2-3h-3s^2)^2 \Big[\boldsymbol{A}(2-3h-3s^2, 2+3s) - \boldsymbol{A}(2-3h-3s^2, 3s)\Big] \\
& + \frac{1}{10s}\Big[2h(13+s-12s^2) - 21h^2 + 2(1+s)^3(3s-2)\Big] \\
& \times \Big[\boldsymbol{A}(3(1+s)^2 - 6h, 3(1+s)) - \boldsymbol{A}(3(1+s)^2 - 6h, 3s-1)\Big] \\
& + \frac{64}{3s}\int_0^1 dx\, x\,[2(s-x)^2 - h]\,\boldsymbol{A}\left(3(s-x)^2 - \frac{3h}{2}, 1-x\right) + (s \to -s) \Bigg\}
\end{aligned}$$

introduce auxiliary function:

$$\boldsymbol{A}(Q,N) = 2\sqrt{Q}\, \arctan\frac{N}{\sqrt{Q}}\,, \ \text{if}\ Q > 0\,, \quad \boldsymbol{A}(Q,N) = \sqrt{-Q}\, \ln\frac{|N+\sqrt{-Q}|}{|N-\sqrt{-Q}|}\,, \ \text{if}\ Q < 0$$





- Resulting energy per particle at 5-loop order takes the form:

$$\bar{E}(k_f) = \Gamma_3{}^2 M\, k_f^{10} \left\{ \frac{54}{35}(11 - 2\ln 2) + \frac{37\pi}{175}\left(\sqrt{3}\ln\frac{k_f}{\lambda} + \zeta_0 + \zeta_1 + \zeta_2\right) \right\} > 0$$

- Numerical values: $\zeta_0 = -1.425$, $\zeta_1 = -5.653$, $\zeta_2 = -4.354 \pm 0.014$
- Modulo $\ln(k_f/\lambda)$-term, the balance reads: $\{14.83 - 7.59\}$
- 2nd order effective 2-body interaction dominates, but gets reduced to about half its size by (genuine) 2nd order three-body contribution
- “Anomalous” 2nd order 3-body diagram vanishes at zero temperature