Accurate Simulation of the Finite Density Lattice Thirring Model

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Motivation

- Calculations at finite density of fermions encounter sign problems
- New ideas are being explored to alleviate or solve sign problems
- 2D lattice Thirring model is an excellent toy model for a sign problem similar to QCD
- This sign problem is solvable using world line methods in the massless limit with open boundary conditions. Accurate simulations are possible.
- With periodic and anti-periodic boundary conditions the sign problem remains but is alleviated in the fermion bag approach.

Plan

- Simulation with the fermion bag algorithm
- Absence of the sign problem

Introduction

We study the lattice Thirring model with the action

$$S = \sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu)\chi_y - \sum_x m\bar{\chi}_x\chi_x - U\sum_{x,\nu} \bar{\chi}_x\chi_x\bar{\chi}_{x+\nu}\chi_{x+\nu}$$

$$D^{KS}_{x,y} = \frac{1}{2}\eta_{x,\nu}e^{\mu\delta_{\nu,0}}\delta_{y,x+\nu} - \frac{1}{2}\eta^{\dagger}_{x,\nu}e^{-\mu\delta_{\nu,0}}\delta_{y,x-\nu}$$

Similar to QCD

- Asymptotically safe
- Dynamically generated fermion mass
- Massless boson but no chiral symmetry breaking [1]
- Sign problem like in QCD

[1] E. Witten, Nucl. Phys. B **145**, 110 (1978).

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Auxiliary Field Representation

$$S = \sum_{x,\nu} \frac{N_F}{g^2} (1 - \cos A_{x,\nu}) + \sum_{x,y} \bar{\chi}_x D_{x,y} \chi_y$$

$$D_{x,y} = m' \delta_{x,y} + \frac{1}{2} \sum_{\nu=0,1} \eta_\nu e^{iA_{x,\nu} + \mu\delta_{\nu,0}} \delta_{x+\nu,y} - \eta_\nu^\dagger e^{-iA_{y,\nu} - \mu\delta_{\nu,0}} \delta_{x-\nu,y}$$

$$U = 0.25 \left(\frac{I_0\left(\frac{N_F}{g^2}\right)}{I_1\left(\frac{N_F}{g^2}\right)} \right)^2 - 0.25, \;\; m = \left(\frac{I_0\left(\frac{N_F}{g^2}\right)}{I_1\left(\frac{N_F}{g^2}\right)} \right) m'$$

$$g = 1 \;\; \rightarrow \;\; U \approx 0.26, \;\; m \approx 1.43 m'$$

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Fermion Bag Representation

Expand partition function

$$Z = \int d\bar{\chi} d\chi e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu)\chi_y + U\bar{\chi}_x\chi_x\bar{\chi}_{x+\nu}\chi_{x+\nu}} \prod_x (1 + m\bar{\chi}_x\chi_x)$$

$$= \sum_{[m]} m^{N_m} \int d\bar{\chi} d\chi \, (\bar{\chi}_x\chi_x)^{m_x} \, e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu)\chi_y - U\bar{\chi}_x\chi_x\bar{\chi}_{x+\nu}\chi_{x+\nu}}$$

$$= \sum_{[m]} m^{N_m} \int_{[f]} d\bar{\chi} d\chi e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu)\chi_y - U\bar{\chi}_x\chi_x\bar{\chi}_{x+\nu}\chi_{x+\nu}}$$

Fermion Bag Representation

Expand partition function

$$Z = \sum_{[m]} m^{N_m} \int_{[f]} d\bar{\chi} d\chi e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu)\chi_y} \left(1 + \sum_{\nu} U \bar{\chi}_x \chi_x \bar{\chi}_{x+\nu} \chi_{x+\nu}\right)$$

$$= \sum_{[d],[m]} U^{N_d} m^{N_m} \int_{[f]} d\bar{\chi} d\chi \left(\bar{\chi}_x \chi_x \bar{\chi}_{x+\nu} \chi_{x+\nu}\right)^{d_{x,\nu}} e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu)\chi_y}$$

$$= \sum_{[d],[m]} U^{N_d} m^{N_m} \int_{[f']} d\bar{\chi} d\chi e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu)\chi_y}$$

$$= \sum_{[d],[m]} U^{N_d} m^{N_m} \det\left(D([f'], \mu)\right)$$

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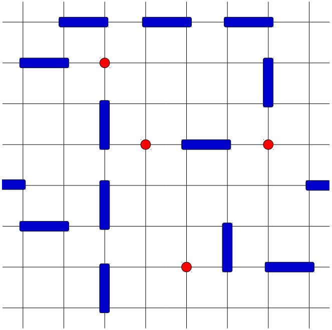
Fermion Bag Representation

- New variables $d_{x,\nu}$ and m_x
- Determinant only over free sites
- Disconnected groups of sites: fermion bags
- Local updates, calculate change in determinant



$$Z = \sum_{[d],[m]} U^{N_d} m^{N_m} \det\left(W([f],\mu)\right)$$

$$= \sum_{[d],[m]} U^{N_d} m^{N_m} \det\left(D(\mu)\right) / \det\left(D^{-1}([o],\mu)\right)$$

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Fermion Bag Representation

Powerlaw scaling of bosonic correlator (at zero mass)

- Zero boson mass without symmetry breaking
- Study the Susceptibility

$$\chi = \frac{U}{V} \sum_{x,y} \langle \bar{\psi}_x \psi_x \bar{\psi}_y \psi_y \rangle \tag{1}$$

- Related to the boson correlator and mass at finite temperature

$$\chi(L_T) = \chi(\infty) - c e^{-m_b L_T} \tag{2}$$

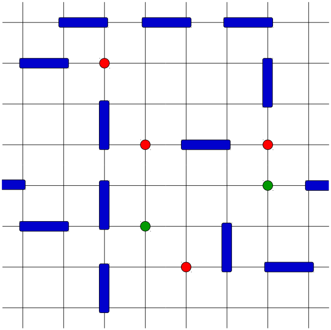
- At zero temperature powerlaw scaling

$$\chi = A L^{2-\eta}, \tag{3}$$

Fermion Bag Representation

- Propagators a bit noisy and require inversions
- Measure χ with a worm update
- Sample with two sources

$$Z_2 = \sum_{xy} \sum_{[d],[m]} U^{N_d} m^{N_m} \int d\bar{\chi} d\chi \, (\bar{\chi}_x \chi_x \bar{\chi}_y \chi_y) \, e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu) \chi_y}$$



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Fermion Bag Representation

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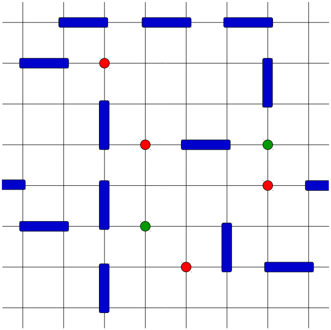

Fermion Bag Representation

- Propagators a bit noisy and require inversions
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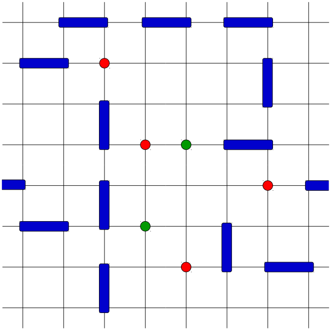
- Sample with two sources

$$Z_2 = \sum_{xy} \sum_{[d],[m]} U^{N_d} m^{N_m}$$

$$\int d\bar{\chi} d\chi \, (\bar{\chi}_x \chi_x \bar{\chi}_y \chi_y) \, e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu) \chi_y}$$



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Fermion Bag Representation

- Propagators a bit noisy and require inversions
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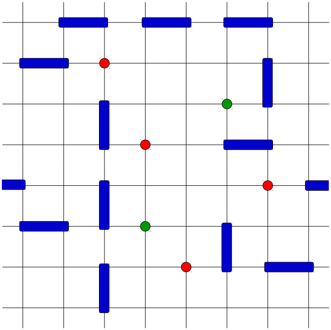
$$Z_2 = \sum_{xy} \sum_{[d],[m]} U^{N_d} m^{N_m} \int d\bar{\chi} d\chi \left(\bar{\chi}_x \chi_x \bar{\chi}_y \chi_y \right) e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu) \chi_y}$$



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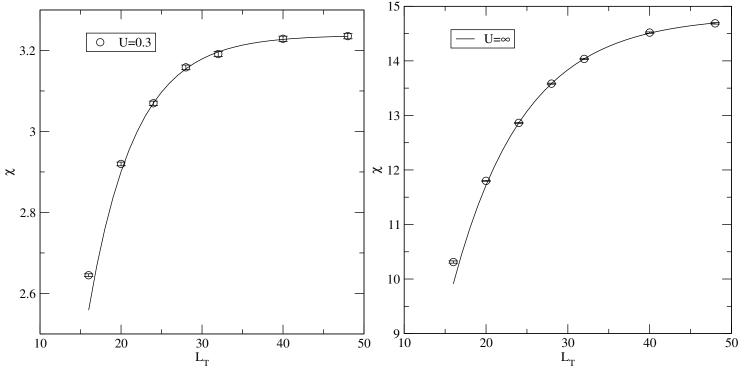
Properties of the Model

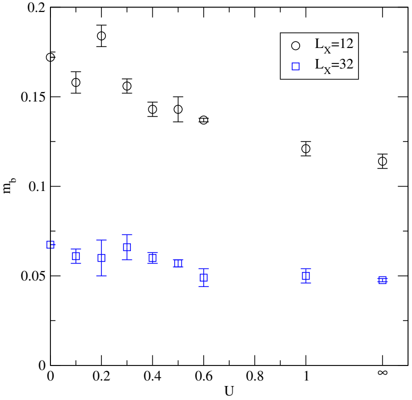
Massless boson:



$$\chi(L_T) = \chi(\infty) - ce^{-m_b L_T}$$

Properties of the Model

Massless boson:



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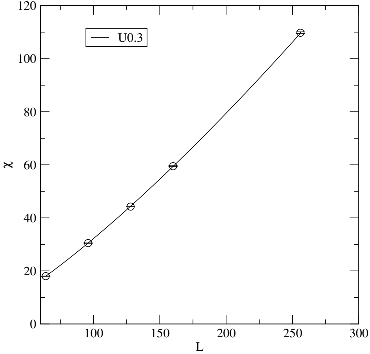
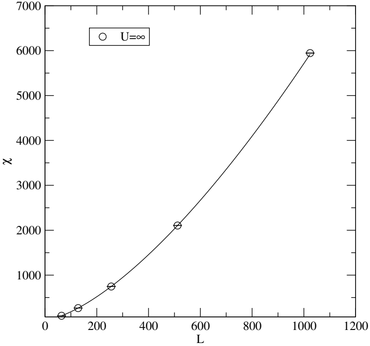
Powerlaw Scaling

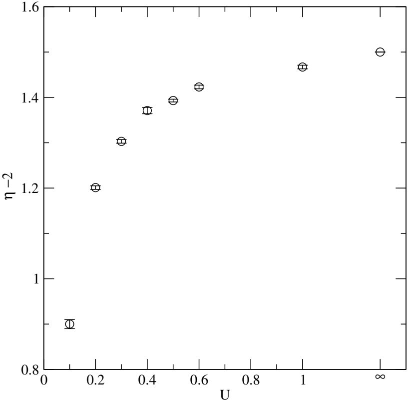




$$\chi = AL^{2-\eta}$$

Properties of the Model

Powerlaw Scaling

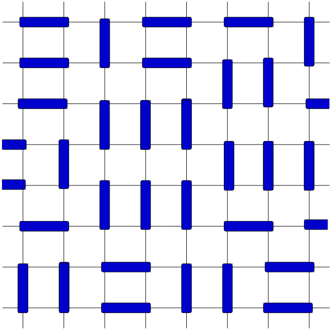


Properties of the Model

Dynamically generated fermion mass:

- At large U limit $m_f = \infty$
- Asymptotically free scaling at small U

$$m_f = C \exp\left(\frac{-2\pi}{b_0 U}\right)$$



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Properties of the Model

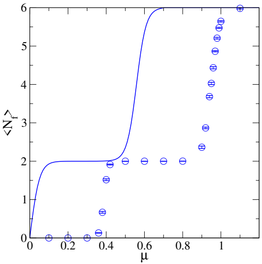
Dynamically generated fermion mass:

- At intermediate U find the value of μ capable of exiting a fermion

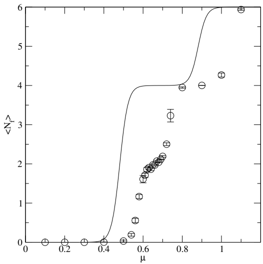
$$\langle N_f \rangle = \left\langle \sum_{x \in S} \frac{\eta_{x,\alpha}}{2} \left[e^{\mu} \bar{\psi}_x \psi_{x+\alpha} - e^{-\mu} \bar{\psi}_{x+\alpha} \psi \right] \right\rangle$$

- With $\mu \neq 0$ we can have a sign problem
- Does not happen with open boundary conditions

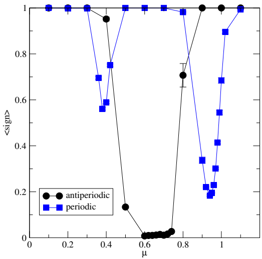
Sign Problem



Periodic



Anti-periodic

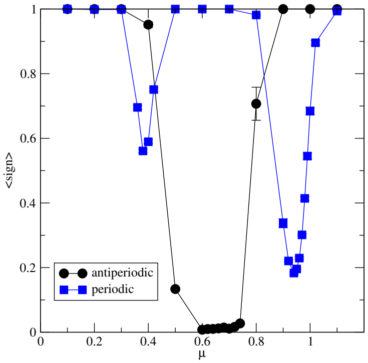


Average sign

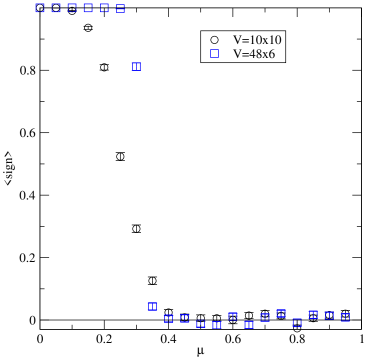
- Small lattice, $L_X = 6$ and $L_T = 48$
- Intermediate coupling $U = 0.3$
- Levels determined by symmetry

Sign Problem

- A quick test with the auxiliary field



Fermion Bag



Auxiliary field

Absence of the Sign Problem

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World Line Representation

No sign problem with:

- open boundary conditions

$$\bar{\chi}_x = \chi_x = 0 \text{ when } x_1 = 0, L_X$$

- $m = 0$ (no monomers)

We show this in the world line formalism

World Line Representation

$$\det\left(W([f],\mu)\right)$$
$$= \prod_{x\in[f]}\left(\int d\bar{\chi}_x d\chi_x\right) e^{-\sum_{x\in[f]}\left(\frac{1}{2}\eta_{x,\nu}e^{\mu\delta_{\nu,0}}\bar{\chi}_x\chi_{x+\nu}-\frac{1}{2}\eta^{\dagger}_{x,\nu}e^{-\mu\delta_{\nu,0}}\bar{\chi}_{x+\nu}\chi_x\right)}$$
$$= \sum_{[l]}\prod_{loop\in l}\left(-\prod_{x,\alpha\in loop} e^{\pm\mu\delta_{\pm\alpha,\hat{t}}}\frac{s_\alpha\eta_{x,\alpha}}{2}\right)$$

$s_\alpha = +1\ (-1)$ for positive (negative) directions α

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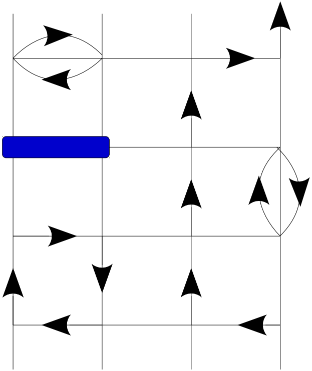
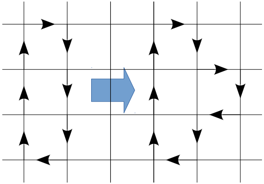
World Line Representation

- Fermion world line variable $l_{x,\nu}$
- Closed loops of fermion world lines

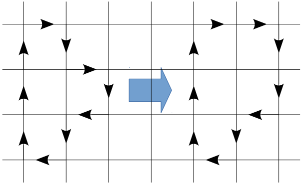


World Line Representation

- Loops can be deformed into each other using two operations:

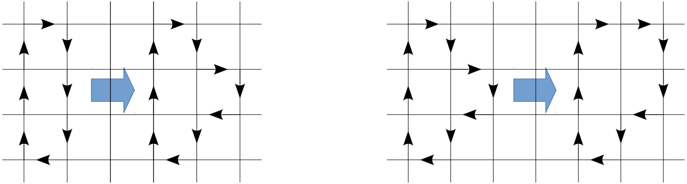


(a) does not change the sign or the volume



(b) changes the sign and the enclosed volume

World Line Representation



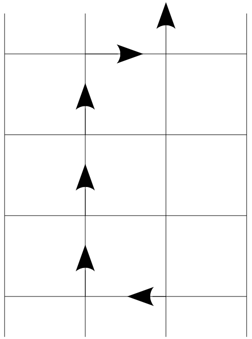
Enumerate all possible loops

- Only even volumes can be enclosed
  $\rightarrow$ only positive signs

World Line Representation

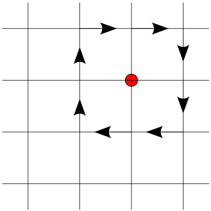
The same idea for time wrapping lines:

- Volume enclosed on both sides
- Boundary conditions are relevant here

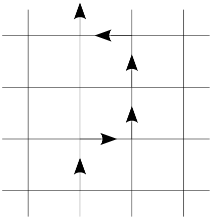


World Line Representation

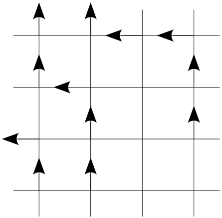
Examples of negative loops:



$m \neq 0$



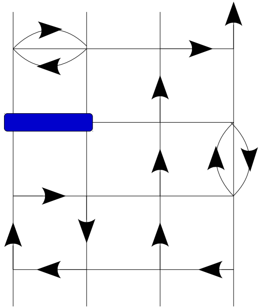
(Anti-)periodic



Anti-periodic

World Line Representation

- Mass allows filling an odd volume
  $\rightarrow$ Sign problem with any boundary conditions
- With other boundary conditions signs arise from wrapping loops
- No sign problem in the $N_f = 0$ sector



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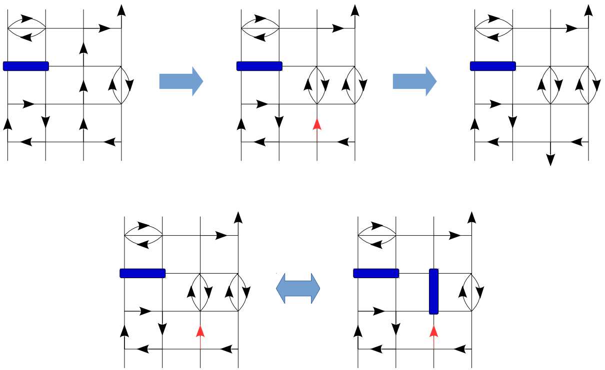
World Line Algorithm

Update with a worm algorithm (m=0)

- Avoid trapping in regions of N_f



- Very easy to implement

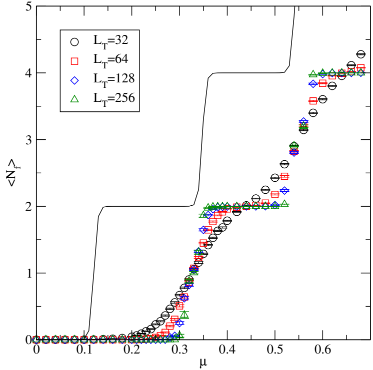
Fermion Number

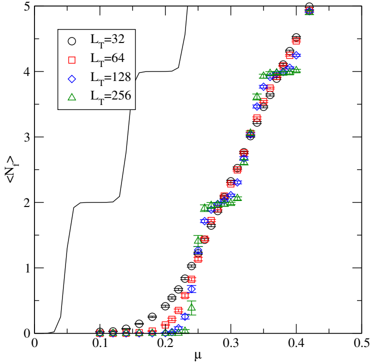
- Fermion number

$$\langle N_f \rangle = \left\langle \sum_{x \in S} \frac{\eta_{x,\alpha}}{2} \left[e^{\mu} \bar{\psi}_x \psi_{x+\alpha} - e^{-\mu} \bar{\psi}_{x+\alpha} \psi \right] \right\rangle$$

$$= \left\langle \sum_{x \in S} l_{x,\hat{t}} - l_{x+\hat{t},-\hat{t}} \right\rangle$$

Fermion Number





Fermion number at $U = 0.3$ and $L_X = 12$ (left) and 32 (right)

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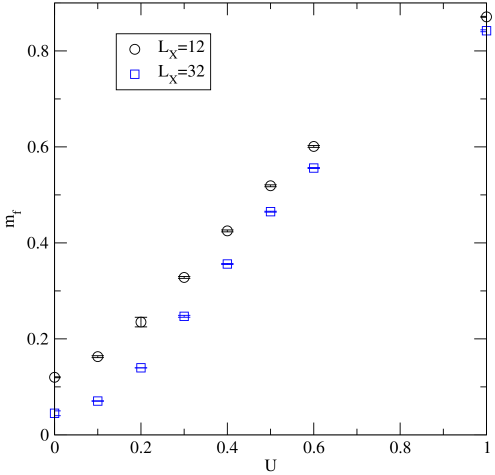
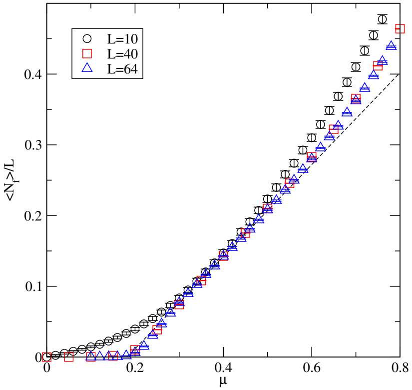
Properties of the Model

Fermion mass



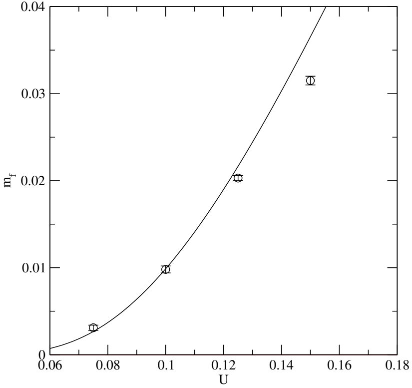
Properties of the Model

Fermion number density on a square lattice



$$\langle N_f \rangle = \max(a\,(\mu - m_f)\,, 0)$$

Results



$$m_f = C \exp\left(\frac{-2\pi}{b_0 U}\right), \quad b_0 = 16, C = 0.49$$

Conclusions

- Efficient simulation of the 2D lattice Thirring model
  - Asymptotically free
  - Powerlaw scaling
  - Zero boson mass
  - Dynamically generated fermion mass
- Significant sign problem with anti-symmetric boundaries
- Sign problem absent with
  - Open boundary conditions
  - Zero fermion mass
- Allows a fast worm algorithm