Sum Rules Involving Higher Electromagnetic Moments

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Motivation

- GDH (Gerasimov – Drell – Hearn) sum rule
  - valid for spin – $^1/_2$ particles
  - relation between cross – section σ and anomalous magnetic moment $\varkappa$

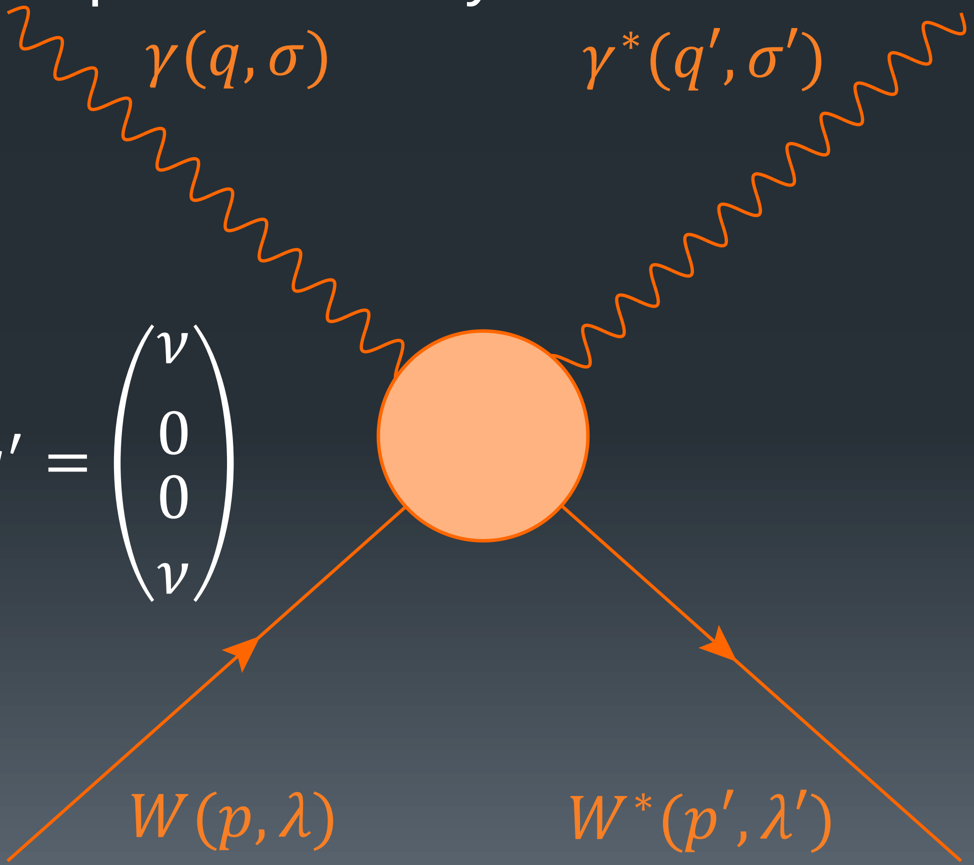
- aim: establish new sum rules
  - valid for spin – 1 particles (W – boson, deuteron)
  - including higher electromagnetic moments

Compton Scattering

- **real Compton scattering:**

$$p^2 = p'^2 = M^2$$
$$q^2 = q'^2 = 0$$

photon helicity fixed: $\sigma = +1$



- **forward limit:**

$$t = (p+q)^2 := 0$$

$$p = p' = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } q = q' = \begin{pmatrix} \nu \\ 0 \\ 0 \\ \nu \end{pmatrix}$$

$$\nu(s) = \frac{s - M^2}{2M}$$

GDH Sum Rule

- **forward scattering amplitude:**

$$T(\nu,\theta = 0) = \vec{\epsilon}^{*} \cdot \vec{\epsilon}\, F_0(\nu) + i\,\nu\,\vec{\sigma} \cdot (\vec{\epsilon}^{*} \times \vec{\epsilon})\, F_1(\nu)$$

- **dispersion relation:**

$$\mathrm{Re}[F_i(\nu)] = \frac{1}{\pi}\int_{-\infty}^{\infty} d\nu' \frac{1}{\nu' - \nu}\, \mathrm{Im}[F_i(\nu')]$$

  - **crossing symmetry:** $F_0(\nu) = F_0(-\nu)$ and $F_1(\nu) = F_1(-\nu)$

$$\mathrm{Re}[F_i(\nu)] = \frac{2}{\pi}\int_{0}^{\infty} d\nu' \frac{\nu\prime}{\nu'^2 - \nu^2}\, \mathrm{Im}[F_i(\nu')]$$

- **optical theorem:**

$$\mathrm{Im}[F_0(\nu)] = \frac{\nu}{8\pi}\left(\sigma_{-1/2} + \sigma_{+1/2}\right)$$

$$\mathrm{Im}[F_1(\nu)] = \frac{1}{8\pi}\left(\sigma_{-1/2} - \sigma_{+1/2}\right)$$

GDH Sum Rule

$$\mathrm{Re}[F_0(\nu)] = \frac{1}{4\pi^2}\int_0^\infty d\nu' \frac{\nu'^2}{\nu'^2 - \nu^2}\left(\sigma_{-1/2} + \sigma_{+1/2}\right)$$

$$\mathrm{Re}[F_1(\nu)] = \frac{1}{4\pi^2}\int_0^\infty d\nu' \frac{\nu'}{\nu'^2 - \nu^2}\left(\sigma_{-1/2} - \sigma_{+1/2}\right)$$

- **low – energy expansion:**
  - $F_0(\nu) = -\frac{e^2}{4\pi M_N} + (\alpha_{E1} + \beta_{M1})\,\nu^2 + \mathcal{O}(\nu^4)$
  - $F_1(\nu) = -\frac{e^2\kappa_N^2}{8\pi M_N^2} + \gamma_0\nu^2 + \mathcal{O}(\nu^4)$

Low – Energy Theorem
[Low, Gell – Mann, Goldberger (1954)]
dipole polarizabilities
higher order polarizabilities

Spin – $^1/_2$ Sum Rules

- **GDH sum rule** (1966):
  - helicity dependent

$$\int_0^\infty d\nu' \frac{1}{\nu'}\left(\sigma_{+1/2} - \sigma_{-1/2}\right) = \frac{e^2 \pi \kappa_N^2}{2M_N^2}$$

- **Forward Spin Polarizability (FSP):**
  - helicity dependent

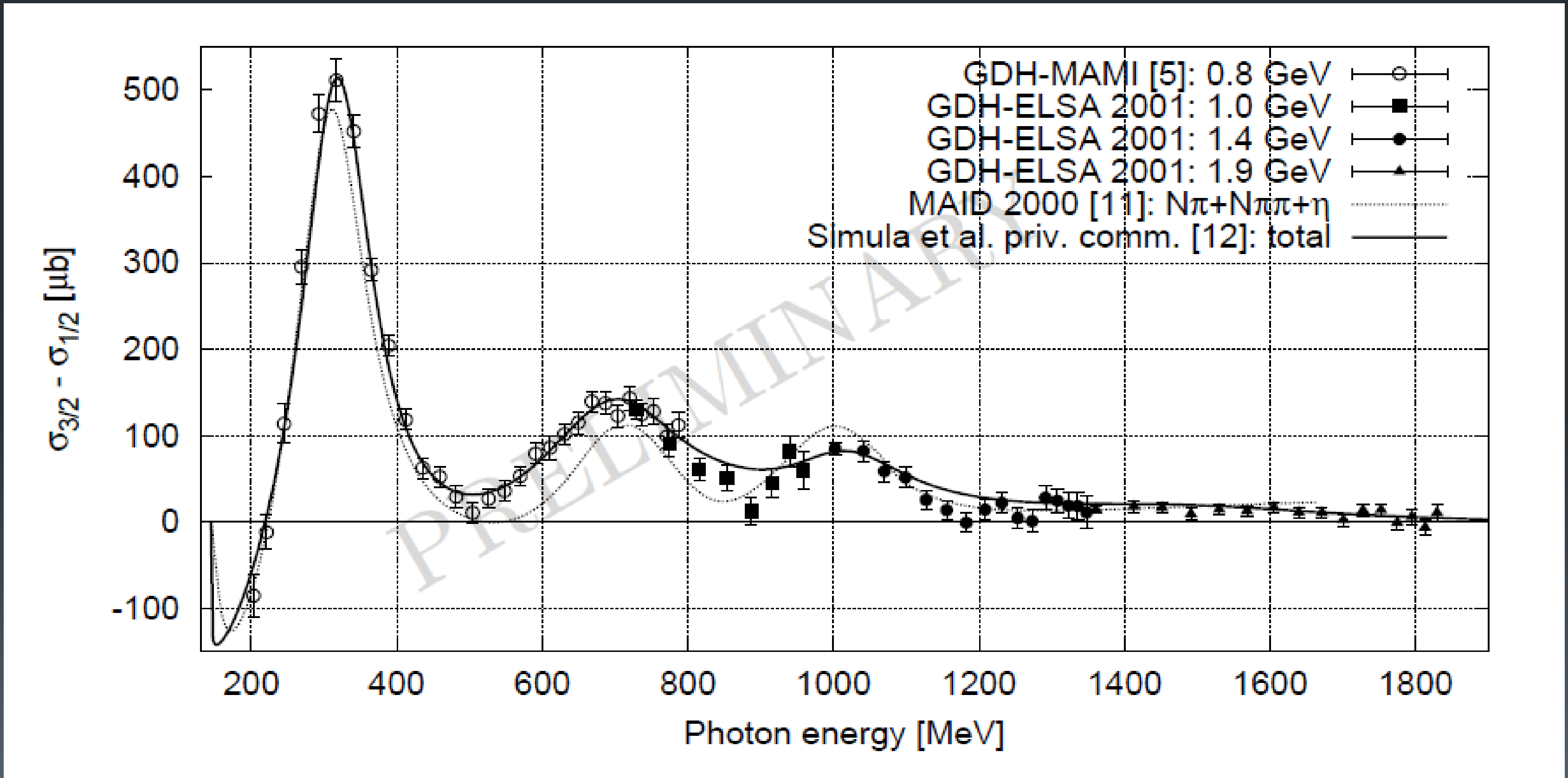
$$\int_0^\infty d\nu' \frac{1}{\nu'^3}\left(\sigma_{-1/2} - \sigma_{+1/2}\right) = 4\,\pi^2\,\gamma_0$$

- **Baldin sum rule** (1960):
  - helicity independent

$$\int_0^\infty d\nu' \frac{1}{\nu'^2}\left(\sigma_{-1/2} + \sigma_{+1/2}\right) = 4\,\pi^2\,(\alpha_{E1} + \beta_{M1})$$

Experimental Test

$$\int_0^\infty d\nu' \, \frac{1}{\nu'} \left(\sigma_{+1/_2} - \sigma_{-1/_2} \right) = \frac{e^2 \, \pi \, \kappa_N^2}{2M_N^2}$$

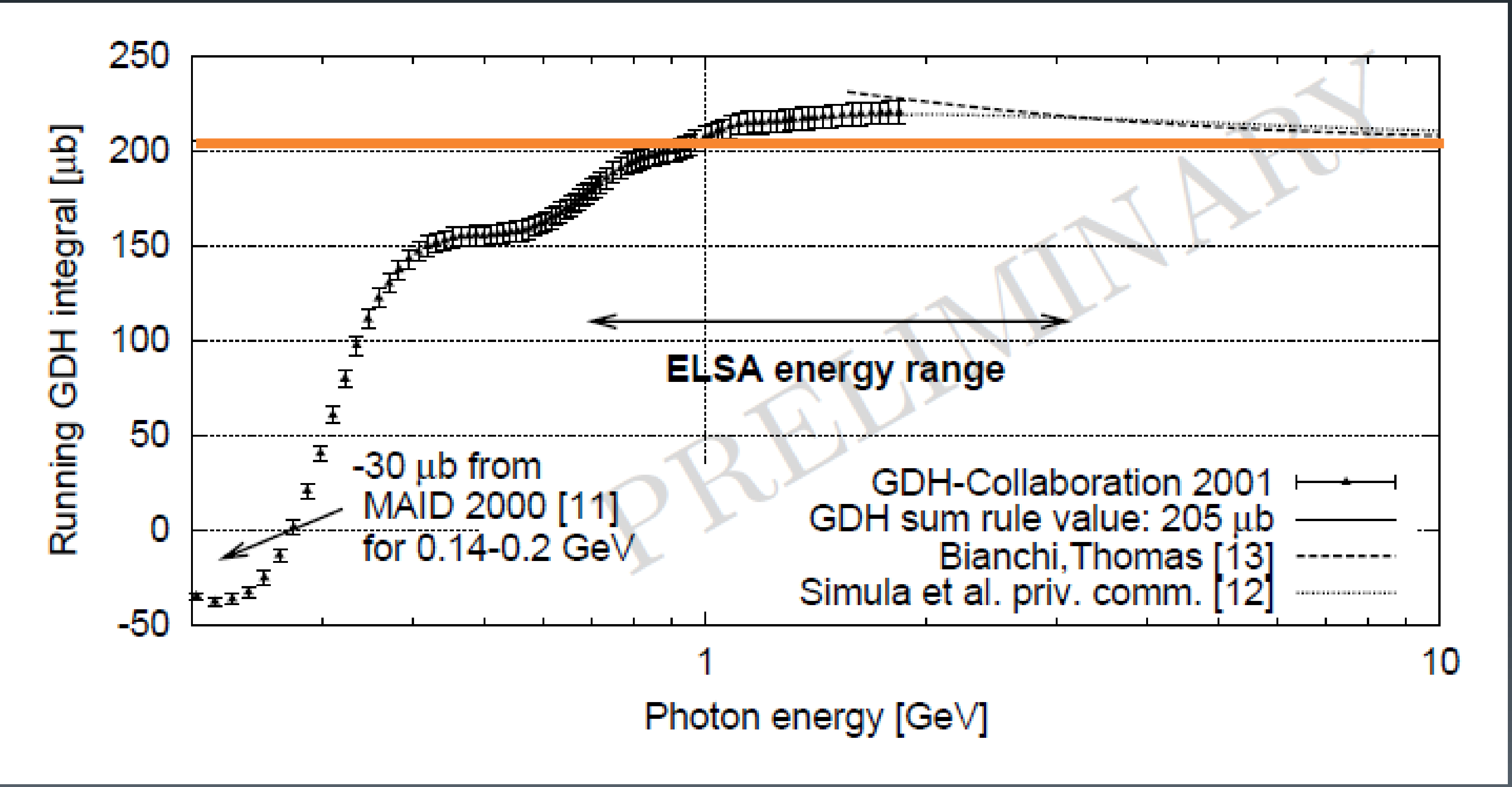


First experimental check of the GHD sum rule, Günter Zeitler for the GDH – Collaboration

Experimental Test

$$\int_0^{\nu} d\nu' \, \frac{1}{\nu'} \left(\sigma_{+1/_2} - \sigma_{-1/_2}\right) = \frac{e^2 \, \pi \, \kappa_N^2}{2M_N^2}$$

running GDH integral



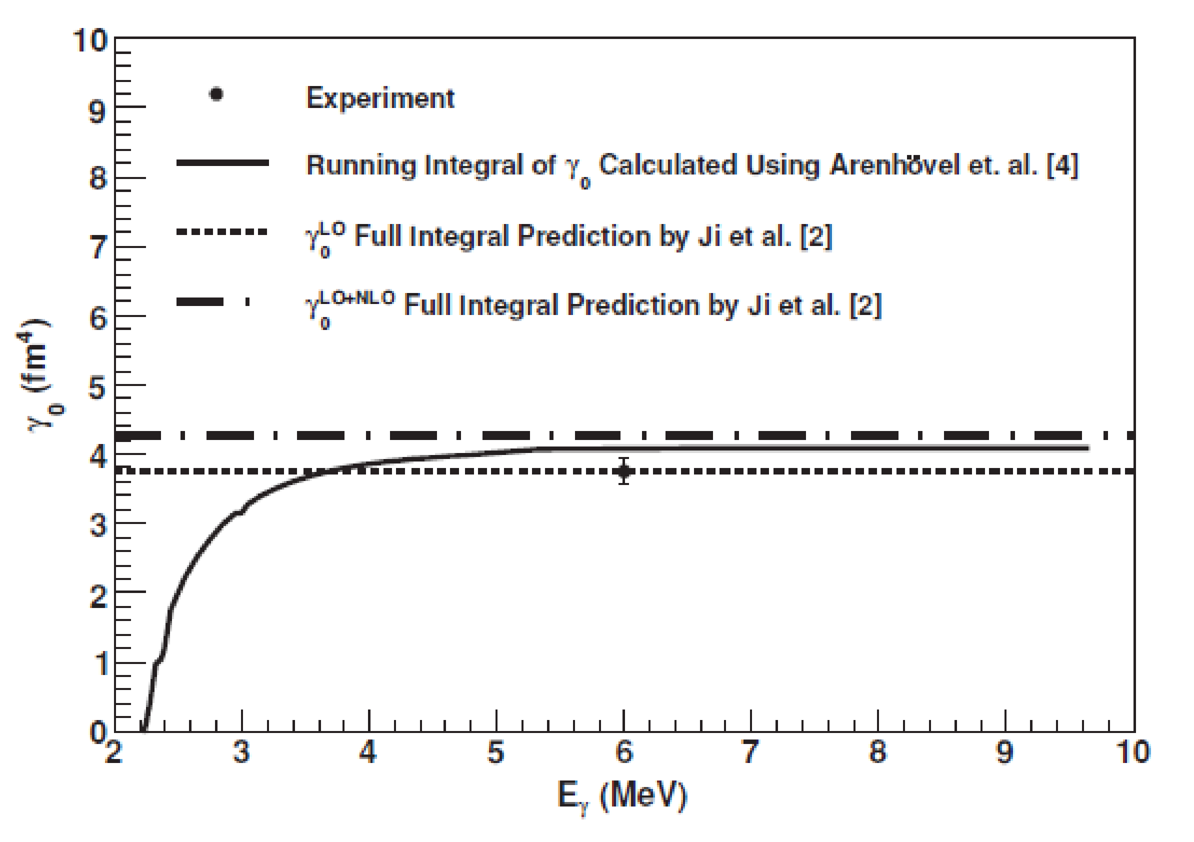
205 μb

First experimental check of the GHD sum rule, Günter Zeitler for the GDH – Collaboration now SPIN – 1 !!!

γ_0 for the Deuteron

assumed the sum rule values are also valid for spin – 1

- **Forward Spin Polarizability:** $\int_{\nu_{th}}^{\infty} d\nu' \frac{1}{\nu'^3}\left(\sigma_{-1/2} - \sigma_{+1/2}\right) = 8\pi^2\gamma_0$



M. W. Ahmed, et al., Phy. Rev. C 77, 044005 (2008)

Sum Rules for Spin – 1

- a particle of arbitrary spin S has in general $2S + 1$ electromagnetic moments
  - gyromagnetic ratio $g = \frac{\mu}{S}\left(\frac{e}{2M}\right)^{-1}$
  - $g = 2$ for all charged pointlike relativistic particles with spin (classically!)

Sum Rules for Spin – 1

- natural electromagnetic moments:$(S = 1)$
  - magnetic dipole moment: $\mu = \frac{e}{M}$
  - electric quadrupole moment: $Q = -\frac{e}{M^2}$
- anomalous electromagnetic moments:
  - ➢ deviation from natural values
  - anomalous magnetic moment: $\varkappa = \frac{M}{e}\mu - 1$
  - anomalous quadrupole moment: $\mathcal{Q} = \frac{1}{2}\left(\frac{M^2}{e}Q + 1\right)$

➢ spin – 1 sum rule should involve also anomalous quadrupole moment $\mathcal{Q}$

- aim: reexamine the sum rules for spin – 1
  and verify them in QED

Sum Rules for Spin – 1

- forward scattering amplitude:

$$T_\lambda(\nu) = \frac{e^2}{M}\sum_{n=0}^{2S} f_n(\nu)\left(\frac{\lambda\nu}{SM}\right)^n$$

$2S+1$
scalar amplitudes $f_0(\nu), \ldots, f_{2S}(\nu)$

$$\overset{S=1}{\Longrightarrow} \frac{e^2}{M}\sum_{n=0}^{2} f_n(\nu)\left(\frac{\lambda\nu}{M}\right)^n = \sum_{n=0}^{2} F_n(\nu)\,(\lambda\nu)^n$$

- spin – 1:
  - Low – Energy Theorem:

$$f_0(\nu) = -1 + 2\frac{\nu^2}{M^2}\varkappa(\varkappa+Q)$$

$$f_1(\nu) = \varkappa^2 + \frac{\nu^2}{4M^2}(\varkappa+Q)^2$$

$$f_2(\nu) = -3\varkappa(\varkappa+Q)$$

Sum Rules for Spin – 1

- GDH sum rule:

$$\frac{e^2}{M^2}\varkappa^2 = \frac{1}{\pi}\int_0^\infty d\nu\ \frac{\sigma_{+1}(\nu) - \sigma_{-1}(\nu)}{\nu}$$

- Forward Spin Polarizability:

$$\boldsymbol{\frac{e^2}{4M^4}(\varkappa + Q)^2} + 2\,\pi\,\gamma_0 = \frac{1}{\pi}\int_0^\infty d\nu\ \frac{\sigma_{+1}(\nu) - \sigma_{-1}(\nu)}{\nu^3}$$

  - anomalous quadrupole moment seems to modify the FSP sum rule

- Quadrupole sum rule (QSR):

$$\boldsymbol{3\,\varkappa(\varkappa + Q)} + \gamma_1 - \gamma_2 = \frac{1}{8\pi}\int_0^\infty d\nu\ \frac{\sigma_{\parallel}(\nu) - \sigma_{\perp}(\nu)}{\nu^3}$$

  - new sum rule

Feynman Rules for Spin – 1 QED

- **W – boson propagator:**

$$\Delta_{\alpha\beta}(l) = -\frac{1}{l^2 - M^2 + i0^+}\left(g_{\alpha\beta} - \frac{l_\alpha l_\beta}{M^2}\right)$$

- **photon propagator:**

$$D_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2 + i0^+}$$

- **3 – point – vertex:** γWW

$$\Gamma^{\alpha\beta\mu}(p,p') = -e\left(g^{\alpha\beta}P^\mu - p'^\beta g^{\alpha\mu} - p^\alpha g^{\beta\mu} - q^\beta g^{\alpha\mu} + q^\alpha g^{\beta\mu}\right)$$

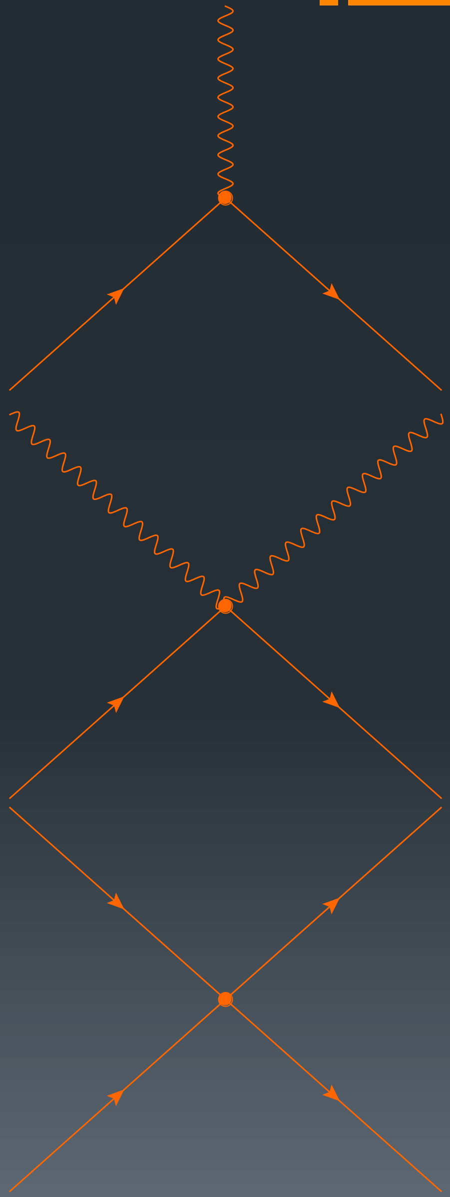
$P = p + p'$ and $q = p' - p$

- **seagull vertex:** $\gamma\gamma WW$

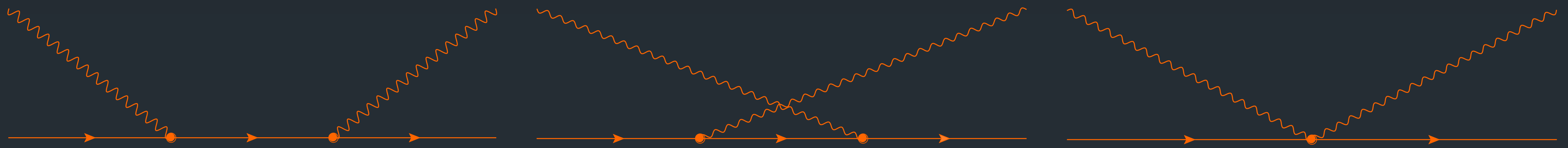
$$\Gamma^{\alpha\beta\mu\nu} = e^2\left(2g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right)$$

- **4 – point – vertex:** $WWWW \rightarrow$ bosonic self – interaction

$$\Gamma^{\alpha\beta\rho\tau} = e^2\left(2g^{\alpha\beta}g^{\rho\tau} - g^{\alpha\rho}g^{\beta\tau} - g^{\alpha\tau}g^{\beta\rho}\right)$$



Tree Level for Spin – 1



$$\sigma_\lambda = \frac{2\pi}{64\pi^2 s}\frac{1}{2}\int_{-1}^{1} dcos(\theta) \sum_{\lambda'\sigma'} \left|T_{\lambda'\sigma',\lambda+1}\right|^2$$

$$\Delta\sigma = \sigma_{+1} - \sigma_{-1}$$
$$= -\frac{e^4}{36\,\pi\,M\,\nu\,(M+2\nu)^3}(15M^3 + 81M^2\nu + 142M\nu^2 + 76\nu^3) + \frac{e^4}{24\,\pi\,M\,\nu^2}(5M+2\nu)\,\mathrm{Log}\left[1+\frac{2\nu}{M}\right]$$

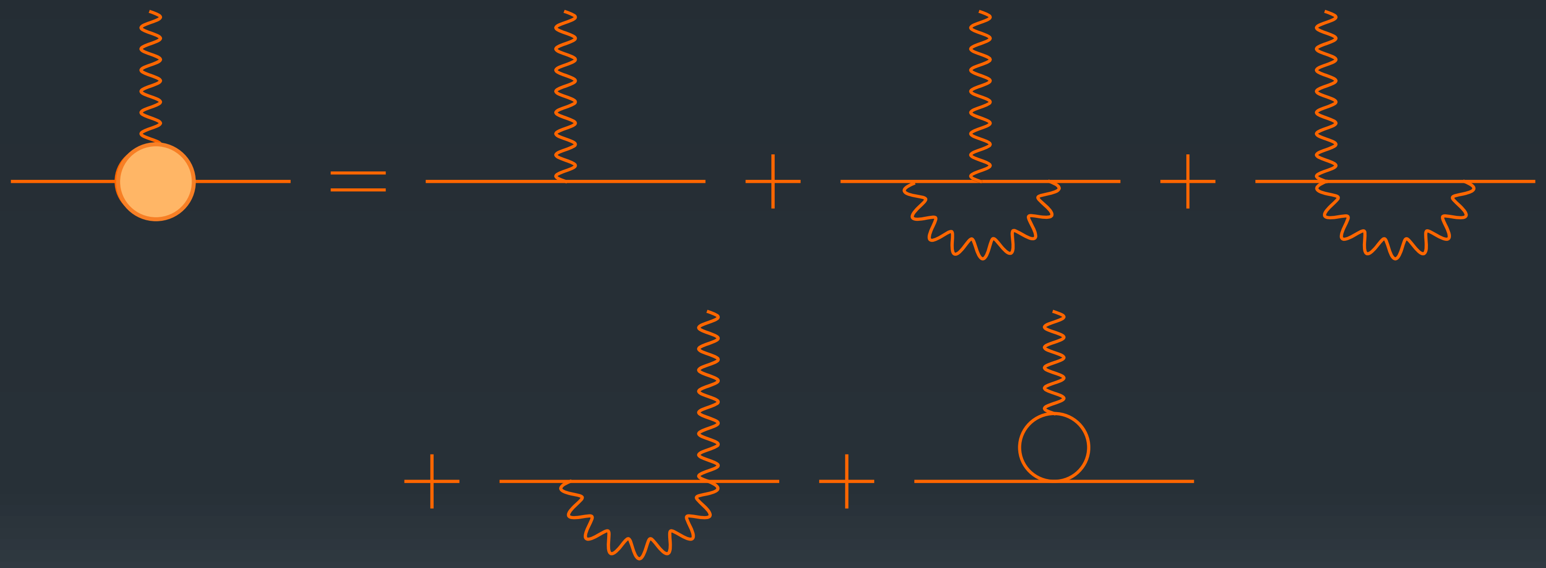
- **GDH sum rule:**

remember: $\alpha = \frac{e^2}{4\pi}$ and $\varkappa \propto \alpha$

$$\frac{e^2}{M^2}\varkappa^2 = 0 + \mathcal{O}(\alpha^3) = \frac{1}{\pi}\int_0^\infty d\nu\,\frac{\Delta\sigma}{\nu} = 0$$

One – Loop Diagrams

- vertex modification:



- propagator modification:

One – Loop Diagrams

$$F_0(\nu) = \frac{171i + 835\pi^2 - 6\,L\,(27i + 79\,\pi^2)}{576\,\pi^4} - \frac{i}{2\,\pi\,M}\,\nu + \frac{\left(-5\,L + 12\,\mathrm{Log}\left[\frac{M}{2\nu}\right] + 10 + 6\,i\,\pi\right)}{12\,\pi^2\,M^2}\,\nu^2 + \mathcal{O}(\nu^3)$$

$$F_1(\nu) = \frac{28 - 15\,L - 45\,i\,\pi + 90\,\mathrm{Log}\left[\frac{2\nu}{M}\right]}{144\,\pi^2\,M} + \frac{2\,i}{3\,\pi\,M^2}\,\nu + \frac{\left(41 - 72\,i\,\pi + 144\,\mathrm{Log}\left[\frac{2\nu}{M}\right]\right)}{48\,\pi^2\,M^3}\,\nu^2 + \mathcal{O}(\nu^3)$$

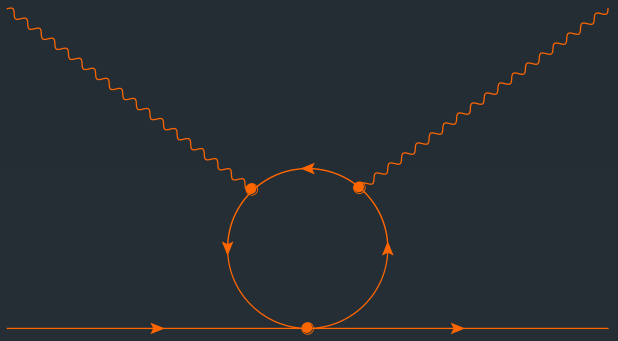
$$F_2(\nu) = \frac{3i(-19 + 18\,L) + 11(-31 + 18\,L)\,\pi^2}{192\,\pi^4\,\nu^2} + \frac{3i}{16\,\pi\,M\,\nu} + \frac{-37 + 20\,L + 8\,i\,\pi + 16\,\mathrm{Log}\left[\frac{M}{2\nu}\right]}{48\,\pi^2 M^2} - \frac{i}{2\,\pi\,M^3}\,\nu + \frac{-2647 + 3270\,i\,\pi + 6540\,\mathrm{Log}\left[\frac{M}{2\nu}\right]}{3600 M^4 \pi^2}\,\nu^2 + \mathcal{O}(\nu^3)$$

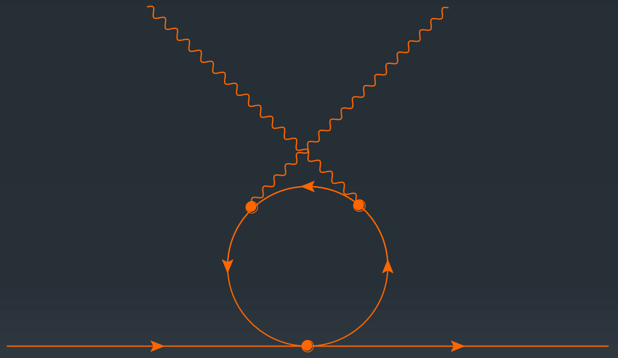
Tadpole Diagrams

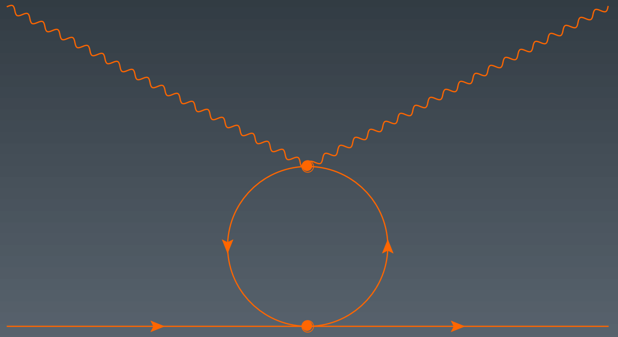
$$F_0(\nu) = \frac{1}{64\,\pi^2}\left(13\,L - \frac{7}{2}\right) - \frac{i}{128\,\pi^4}\left(13\,L - \frac{7}{2}\right) - \frac{1}{48\,\pi^2\,M^2}(5+L)\,\nu^2$$

$$F_1(\nu) = 0$$

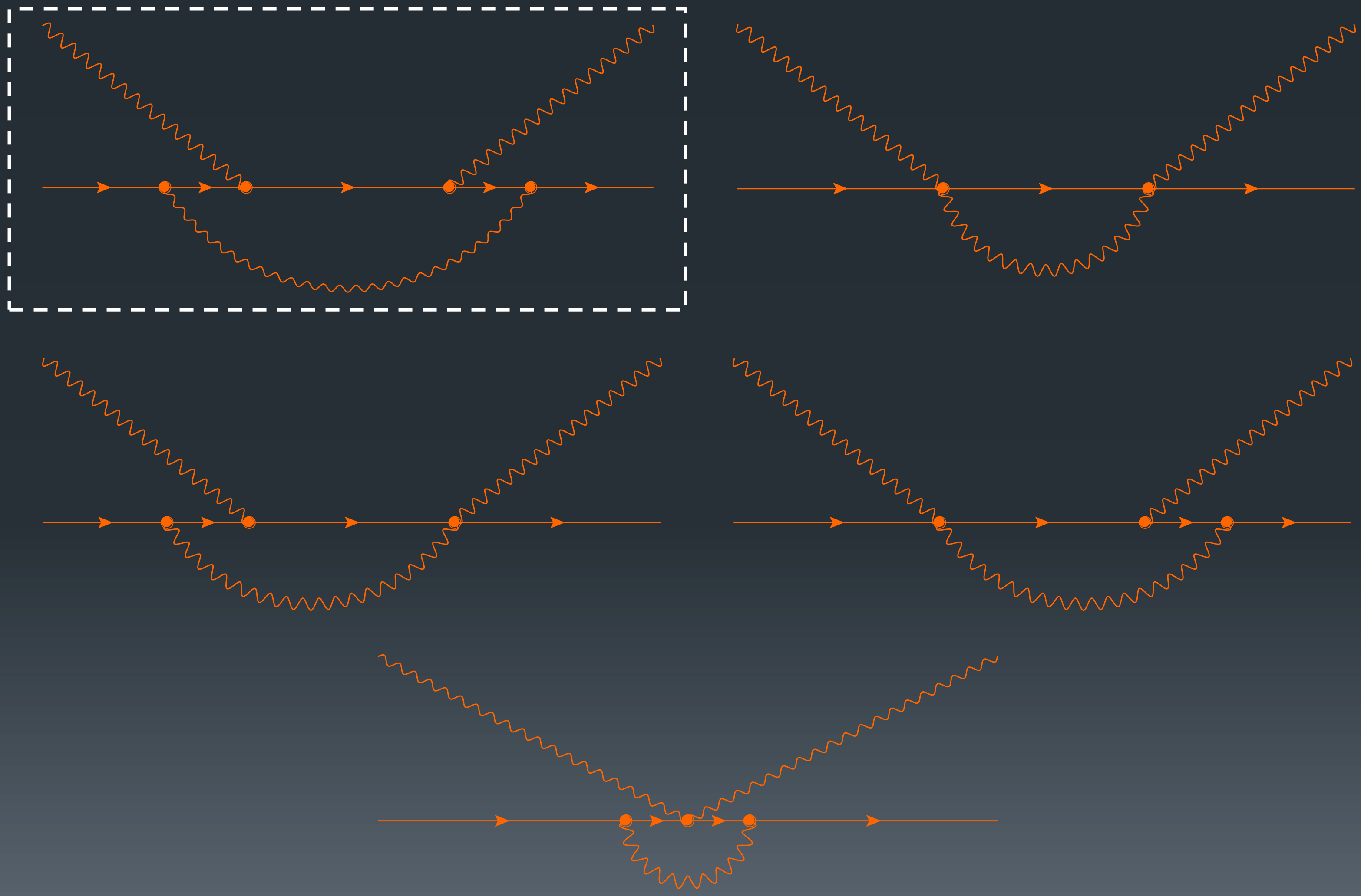
$$F_2(\nu) = \frac{1}{64\,\pi^2}\left(L + \frac{5}{2}\right)\frac{1}{\nu^2} - \frac{i}{128\,\pi^4}\left(L + \frac{5}{2}\right)\frac{1}{\nu^2} + \frac{1}{48\,\pi^2\,M^2}(5+L)$$







Irreducible Diagrams



Conclusion

- GDH sum rule is valid for $\mathcal{O}(\alpha^2)$
- other sum rules will be tested in the end of the calculations
  - Forward Spin Polarizability:

$$\frac{e^2}{4\,M^4}(\varkappa + \mathcal{Q})^2 + 2\,\pi\,\gamma_0 = \frac{1}{\pi}\int_0^\infty d\nu\;\frac{\sigma_{+1}(\nu) - \sigma_{-1}(\nu)}{\nu^3}$$

  - Quadrupole sum rule (QSR):

$$3\,\varkappa\,(\varkappa + \mathcal{Q}) + \gamma_1 - \gamma_2 = \frac{1}{8\pi}\int_0^\infty d\nu\;\frac{\sigma_{\parallel}(\nu) - \sigma_{\perp}(\nu)}{\nu^3}$$