

STATISTICAL CLUSTERS

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1. THE STATISTICAL MODEL: CLUSTERS
2. TWO-PARTICLE CORRELATIONS
3. DECAY OF A STATISTICAL CLUSTER
4. CLUSTERS WITH TRANSVERSE MOMENTUM
5. CLUSTER AS A JET
6. MULTIPARTICLE CORRELATIONS
7. CONCLUSIONS, OUTLOOK, QUESTIONS

STATISTICAL MODEL (as I try to understand it)

- 1. STATISTICAL MODEL: HADRONIZATION PROCEEDS THROUGH FORMATION OF INDEPENDENT CLUSTERS WHICH DECAY STATISTICALLY INTO OBSERVED HADRONS (BECATTINI).**
- 2. ALTHOUGH DISTRIBUTION OF CLUSTERS IN MOMENTUM SPACE IS NOT FIXED BY THE MODEL, THIS APPROACH ALLOWS TO EVALUATE RATIOS OF VARIOUS PARTICLE YIELDS AND EVEN TRANSVERSE MOMENTUM SPECTRA. THE RESULTS AGREE WELL WITH DATA.**
- 3. SINCE THIS IDEA SEEMS TO WORK EVERYWHERE, IT LOOKS THAT CLUSTERS REPRESENT A UNIVERSAL FEATURE OF HADRONIZATION. THEREFORE IT IS INTERESTING (URGENT?) TO STUDY THEIR STRUCTURE AND PROPERTIES.**

CLUSTERS AND CORRELATIONS

1. A VERY GENERAL CONSEQUENCE OF THE CLUSTER FORMATION IS THE PRESENCE OF *MULTIPARTICLE CORRELATIONS*.
2. MORE: THE STRUCTURE AND PROPERTIES OF CLUSTERS ARE REFLECTED IN THE STRUCTURE AND PROPERTIES OF CORRELATIONS.
3. THEREFORE, IN SEARCH OF THE DYNAMICAL ORIGIN OF THE STATISTICAL MODEL IT IS NECESSARY TO STUDY CORRELATIONS BETWEEN PRODUCED PARTICLES.
4. PROBABLY THE MOST INTERESTING QUESTION IS THAT OF *UNIVERSALITY*.

TWO-PARTICLE CORRELATIONS

1. EVEN IF PARTICLES FROM ONE CLUSTER ARE EMITTED INDEPENDENTLY, THEY ARE CORRELATED IN MOMENTUM SPACE.

TAKE A CLUSTER AT A POSITION Y IN RAPIDITY. CONSIDER TWO PARTICLES EMITTED FROM THIS CLUSTER AT POSITIONS y_1 and y_2 . THE DISTRIBUTION OF y_1, y_2 IS

$$\rho(y_1, y_2) = \int dY F(Y) f(y_2 - Y) f(y_1 - Y) \quad (1)$$

TAKING GAUSSIANS: $F = e^{-Y^2/\Omega^2}$; $f = e^{-(y-Y)^2/\Delta^2}$ WE OBTAIN

$$\rho(y_1, y_2) \sim e^{-(y_1 - y_2)^2/2\Delta^2} e^{-(y_1 + y_2)^2/2(\Delta^2 + 2\Omega^2)} \quad (2)$$

ONE SEES THAT DISTRIBUTION IN $(y_1 - y_2)$ MEASURES THE WIDTH OF THE CLUSTER.

STATISTICAL CLUSTERS

NEGLECTING QUANTUM STATISTICS, THE DECAY OF THE CLUSTER CENTERED AT $\vec{P} = 0$ IS DESCRIBED BY

$$\rho(\vec{p}) d^3 p / E \sim e^{-E/T} d^3 p / E = e^{-E/T} d^2 p_{\perp} dy \quad (3)$$

ASSUME THE CLUSTER MOVES IN TRANSVERSE DIRECTION WITH VELOCITY V . ONE CAN EVALUATE VARIOUS DISTRIBUTIONS OF EMITTED PARTICLES:

$$\begin{aligned} \rho(p_{\perp}) &\sim K_0 [\gamma m_{\perp} / T] I_0[\gamma V p_{\perp} / T]; \\ \rho(y) &\sim \int p_{\perp} dp_{\perp} e^{-\gamma m_{\perp} \cosh y / T} I_0[\gamma V p_{\perp} / T]. \end{aligned} \quad (4)$$

WITH INCREASING V , $\rho(p_{\perp})$ BROADENS, WHILE $\rho(y)$ GETS NARROWER.

AT $V = 0$:
$$\rho(y) \sim \frac{1 + (m/T) \cosh y}{\cosh^2 y} e^{-(m/T) \cosh y}$$

RAPIDITY CORRELATIONS 1

CONSIDER TWO PARTICLES EMITTED FROM A CLUSTER AT REST (NO TRANSVERSE MOMENTUM) THE RAPIDITY DISTRIBUTION IS THE PRODUCT OF THE SINGLE-PARTICLE DISTRIBUTIONS. THEREFORE THE DISTRIBUTION OF THE DIFFERENCE

$y_- = (y_1 - y_2)/2$ IS

$$D(y_-) = \int dy_+ \rho(y_1) \rho(y_2) = \int dy_+ \rho(y_+ + y_-) \rho(y_+ - y_-) \quad (5)$$

WITH

$$\rho(y) \sim \frac{1 + (m/T) \cosh y}{\cosh^2 y} e^{-(m/T) \cosh y} \quad (6)$$

THE RESULT DOES NOT DEPEND ON THE POSITION OF THE CLUSTER BUT DEPENDS ON THE RATIO m/T .

RAPIDITY CORRELATIONS 2

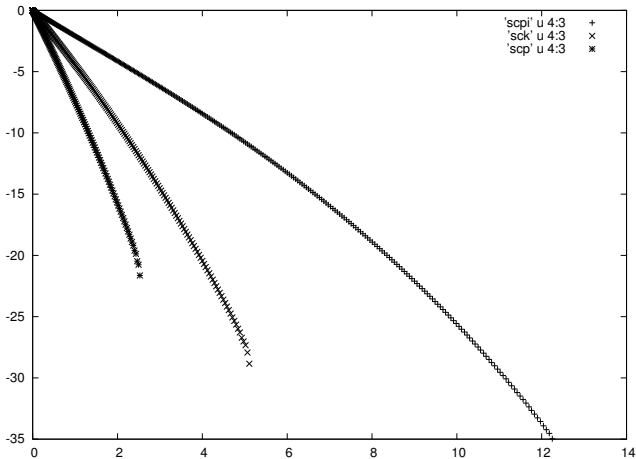


Figure: Distribution of $y_- = (y_1 - y_2)/2$ from decay of a statistical cluster at rest with $T = 160$ MeV, plotted versus y_-^2 . The three lines correspond to pion, kaon and proton mass (proton is steepest).

RAPIDITY CORRELATIONS 3: CLUSTERS MOVING IN TRANSVERSE DIRECTION

CONSIDER A CLUSTER MOVING IN TRANSVERSE DIRECTION WITH VELOCITY v AND THE LORENTZ FACTOR γ .

$$\rho(y)dy = dy \frac{Am^2}{B^2} [1 + 1/B] e^{-B} \quad (7)$$

$$A = (m/T) \sqrt{1 + \gamma^2 v^2} \cosh(y - Y);$$
$$B = (m/T) \sqrt{1 + (1 + \gamma^2 v^2) \sinh^2(y - Y)} \quad (8)$$

AGAIN, THE DISTRIBUTION OF THE DIFFERENCE

$y_- = (y_1 - y_2)/2$ IS

$$\rho(y_-) = \int dy_+ \rho(y_1) \rho(y_2) = \int dy_+ \rho(y_+ + y_-) \rho(y_+ - y_-) \quad (9)$$

RAPIDITY CORRELATIONS 4

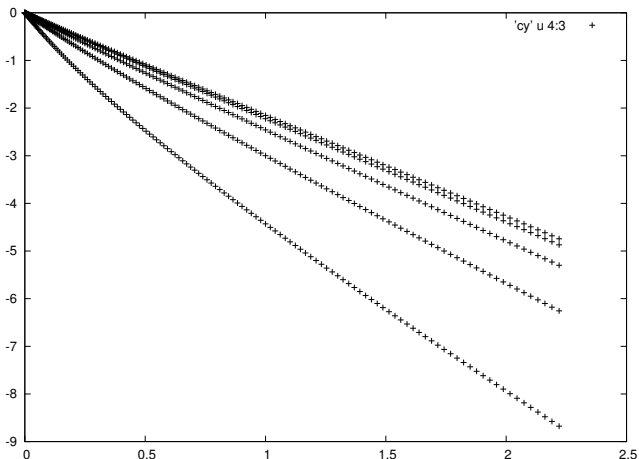


Figure: Distribution of $y_- = (y_1 - y_2)/2$, plotted vs y_+ , from the pion decay of a statistical cluster moving in transverse direction with velocity $v = 0., 0.2, 0.4, 0.6, 0.8$. $T = 160$ MeV. The slope increases with increasing v .

RAPIDITY CORRELATIONS 5

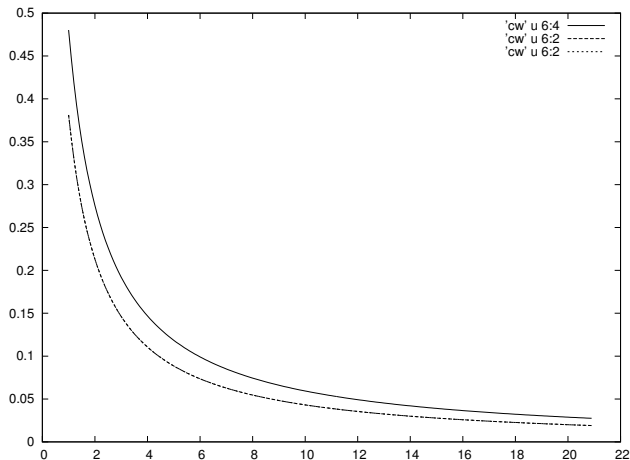


Figure: The width of rapidity distribution in cluster decay versus transverse Lorentz factor. Lower line: $\langle |y| \rangle$; Upper line: $\sqrt{\langle y^2 \rangle}$

P_t CORRELATIONS 0

SIMILAR ARGUMENT CAN BE APPLIED TO EVALUATE THE CORRELATIONS BETWEEN PARTICLE MOMENTA ORTHOGONAL TO THE CLUSTER VELOCITY.

WE HAVE

$$\frac{D(p)}{dp_t^2} = \int p_+ d\phi = d\phi K_0(\gamma m_+) K_0(\gamma m_-) \quad (10)$$

WHERE

$$\vec{p}_\pm = \vec{p}_1 \pm \vec{p}_2; \quad p_\pm = |\vec{p}_\pm|; \quad m_\pm^2 = p_+^2 + p_-^2 \pm 2p_+p_- \cos \phi \quad (11)$$

**AND ϕ IS THE ANGLE BETWEEN \vec{p}_1 AND \vec{p}_2 .
THE RESULTS ARE SHOWN IN THE NEXT FEW SLIDES.**

P_t CORRELATIONS 1

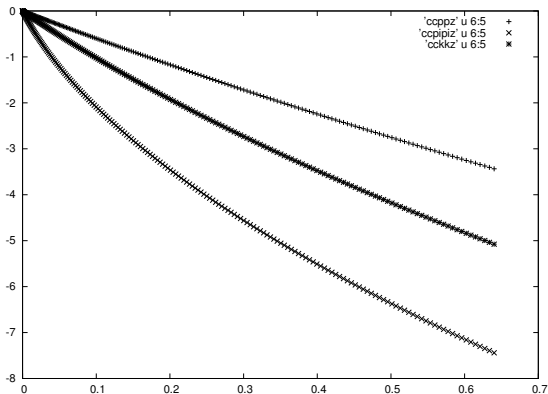


Figure: Correlation function in transverse momentum (in logarithmic scale) plotted vs $|\vec{p}_1 - \vec{p}_2|^2$. Upper line: pp ; lower line: kk ; lowest line: $\pi\pi$

P_t CORRELATIONS 2

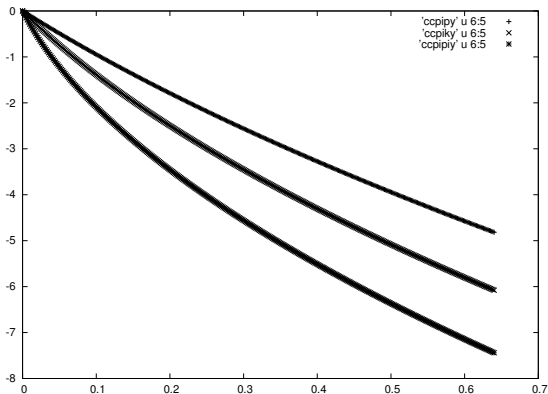


Figure: Correlation function in transverse momentum (in logarithmic scale) plotted vs $|\vec{p}_1 - \vec{p}_2|^2$. Upper line: $p\pi$; lower line: $k\pi$; lowest line: $\pi\pi$

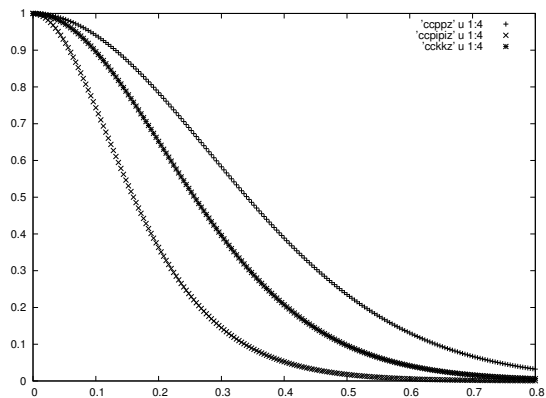


Figure: Correlation function in transverse momentum plotted vs $|\vec{p}_1 - \vec{p}_2|$. Upper line: pp ; lower line: kk ; lowest line: $\pi\pi$

P_t CORRELATIONS 4

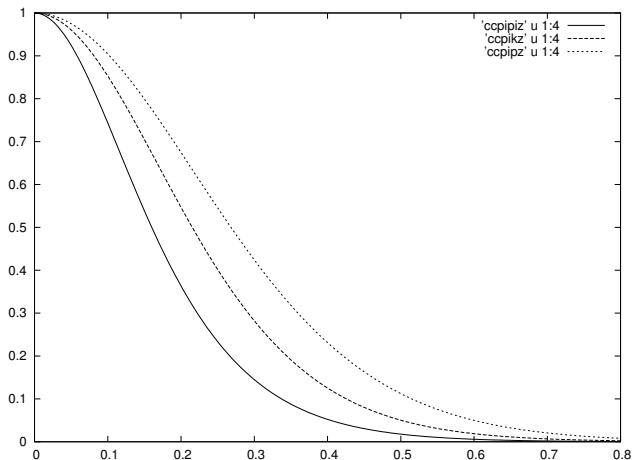


Figure: Correlation function in transverse momentum plotted vs $|\vec{p}_1 - \vec{p}_2|$. Upper line: $p\pi$; lower line: $k\pi$; lowest line: $\pi\pi$

JET AS A STATISTICAL CLUSTER

CONSIDER A CLUSTER AT RAPIDITY = 0, MOVING IN THE TRANSVERSE DIRECTION WITH TRANSVERSE ENERGY DISTRIBUTED ACCORDING TO POWER LAW:

$$dw = (k - 1) \left(\frac{E_c}{M_c} \right)^{-k} d \left(\frac{E_c}{M_c} \right) = (k - 1) \gamma^{-k} d\gamma; \quad \int dw = 1. \quad (12)$$

THE DISTRIBUTION OF TRANSVERSE MOMENTUM OF PARTICLES FROM CLUSTER DECAY IS THEN

$$\frac{dn}{dm_{\perp}^2} \sim \int_1^{\infty} d\gamma \gamma^{-k} I_0[\gamma V p_{\perp} / T] K_0[\gamma m_{\perp} / T] \quad (13)$$

IT TURNS OUT THAT THIS DISTRIBUTION IS VERY CLOSE TO THE TSALLIS DISTRIBUTION, AS SEEN IN THE NEXT SLIDE.

p_{\perp} DISTRIBUTION

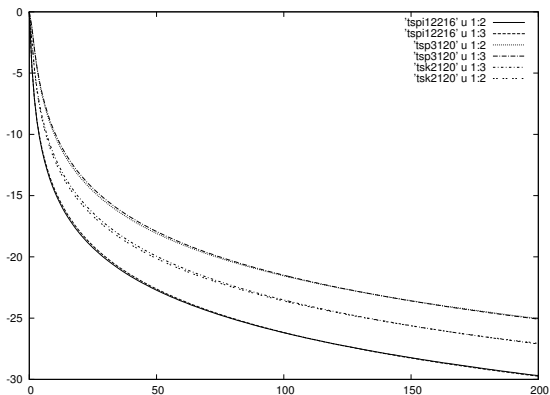


Figure: Transverse momentum distribution dn/dp_{\perp}^2 of a statistical cluster for $p_{\perp} \leq 200$ GeV. $T = 160$ MeV. Distribution of the cluster energy is taken to be $dN/dE \sim 1/E^k$ with $k = 4$. The result is compared with the Tsallis distribution $\sim 1/[1 + (m_{\perp} - m)/\kappa T_{ts}]^{\kappa}$. Pions: with $\kappa = 5.16$ and $T_{ts} = 122$ MeV. Kaons: $\kappa = 5.20$ and $T_{ts} = 200$ MeV. Protons: $\kappa = 5.20$ and $T_{ts} = 310$ MeV. The Tsallis formula is almost perfectly recovered

p_{\perp} DISTRIBUTION

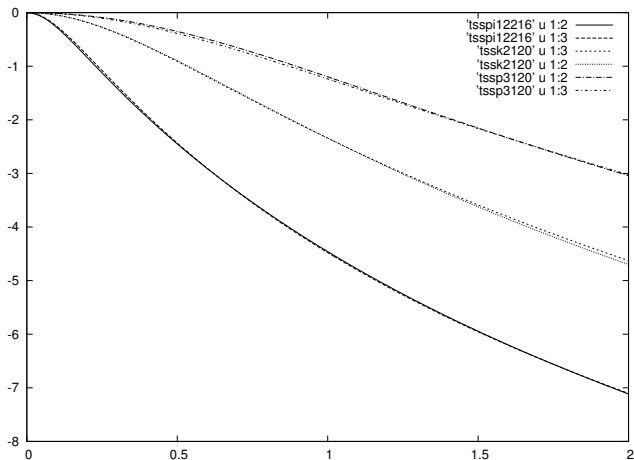


Figure: The same as in Fig. 8 in the region $p_{\perp} \leq 2$ GeV.

CONCLUSIONS AND OUTLOOK

- 1. IN MY OPINION, FORMATION OF CLUSTERS PROVIDES THE BEST ANSWER TO THE QUESTION WHY THE STATISTICAL MODEL WORKS ALMOST EVERYWHERE.**
- 2. IT IS ENCOURAGING THAT THE OBSERVED (TSALLIS) TRANSVERSE MOMENTUM DISTRIBUTION CAN BE RECONSTRUCTED USING THIS IDEA.**
- 3. IF CLUSTERS DO EXIST, THERE ARE SIMPLE CONSEQUENCES FOR MULTI-PARTICLE SPECTRA. MOST INTERESTING ARE (i) TWO PARTICLE CORRELATIONS BOTH IN LONGITUDINAL AND IN TRANSVERSE DIRECTIONS AND (ii) MULTIPLICITY DISTRIBUTIONS IN VARIOUS RAPIDITY BINS.**
- 4. SUCH MEASUREMENTS SHOULD ALLOW TO LEARN ABOUT PROPERTIES OF THE CLUSTERS.**

QUESTIONS

(i) ALL ARGUMENTS I GAVE IGNORE THE ENERGY CONSERVATION. THIS CAN BE JUSTIFIED (?) IF CLUSTERS DO NOT HAVE A WELL-DEFINED MASS BUT RATHER MASS DISTRIBUTION OF THE EXPONENTIAL TYPE:

$$\int dE \delta \left[E - \sum_i E_i \right] e^{-E/T} = e^{-\sum_i E_i/T} \quad (14)$$

(ii) IF CLUSTERS ARE TO BE TREATED SERIOUSLY, ONE SHOULD ASK QUESTIONS ABOUT THEIR SIZE IN CONFIGURATION SPACE. THIS IS (AS FAR AS I KNOW) UNTOUCHED PROBLEM. AGAIN, I FEEL THAT THE CLUSTERS' VOLUME MAY HAVE A RATHER BROAD DISTRIBUTION. NOTE THAT IT MAY HAVE CONSEQUENCES FOR PARTICLE RATIOS.

(iii) THE FUNDAMENTAL QUESTION ABOUT FAST EQUILIBRATION REMAINS OPEN (AT LEAST FOR ME).

APPENDIX 1

We start from the decay of a cluster centered at $\vec{P} = 0$:

$$\rho(\vec{k})d^3k/\epsilon \sim e^{-\epsilon/T}d^3k/\epsilon \quad (15)$$

If this distribution is regarded from a system moving with transverse velocity V in, say, y -direction, we have

$$\begin{aligned} \rho(\vec{p})d^3p/E &\sim e^{-\gamma[Vp_y+E]/T}d^2p_\perp dy = \\ &= e^{-\gamma[Vp_\perp \cos\phi + m_\perp \cosh y]/T}p_\perp dp_\perp d\phi dy \end{aligned} \quad (16)$$

Integration over ϕ gives

$$\rho(p_\perp, y)p_\perp dp_\perp dy = 2\pi I_0[\gamma Vp_\perp/T]e^{-\gamma m_\perp \cosh y/T}p_\perp dp_\perp dy \quad (17)$$

Also integration over y can be explicitly performed:

$$\rho(p_\perp)p_\perp dp_\perp = 2\pi I_0[\gamma Vp_\perp/T]K_0[\gamma m_\perp/T]p_\perp dp_\perp \quad (18)$$

where we have used the identities

$$\int_0^{2\pi} e^{-z \cos\phi} d\phi = I_0(z); \quad \int_0^\infty e^{-z \cosh y} dy = K_0(z) \quad (19)$$

APPENDIX 2

Denoting

$$v_m^{(k)} = \int dY F_m(Y) [\bar{n} g(\delta, Y)]^k \quad (20)$$

we obtain

$$F_1 = \langle n \rangle = \sum_{m=-N}^N v_m^{(1)} \quad (21)$$

$$F_2 = \sum_{m=-N}^N \left\{ v_m^{(2)} - [v_m^{(1)}]^2 \right\} + [F_1]^2 \quad (22)$$

$$\begin{aligned} F_3 &= \sum_{i=-N}^N v_i^{(3)} + 3 \sum_i v_i^{(2)} \sum_{j \neq i} v_j^{(1)} + \sum_{i \neq j \neq k} v_i^{(1)} v_j^{(1)} v_k^{(1)} = \\ &= \sum_{i=-N}^N \left\{ v_i^{(3)} + 2[v_i^{(1)}]^3 - 3v_i^{(2)} v_i^{(1)} \right\} + 3F_2 F_1 - 2F_1^3 \quad (23) \end{aligned}$$

MULTI-PARTICLE CORRELATIONS

SELECT A BIN IN RAPIDITY OF SIZE δ . CONSIDER CONTRIBUTION TO THIS BIN FROM ONE CLUSTER AT A POSITION Y DISTRIBUTED WITH $F(Y)$. IF PARTICLES ARE EMITTED INDEPENDENTLY (POISSON), THE GENERATING FUNCTION OF THE DISTRIBUTION IN THE BIN δ IS

$$\Phi(z, \delta) \equiv \sum_n P(n)z^n = \int_{-\infty}^{\infty} dY F(Y) e^{-\bar{n}g(\delta, Y)(z-1)} \quad (24)$$

WITH

$$g(\delta, Y) = \frac{\int_{\delta} dy' f(y' - Y)}{\int du f(u)} \quad (25)$$

WHERE $f(u)$ IS THE DECAY DISTRIBUTION AND \bar{n} IS THE AVERAGE MULTIPLICITY IN CLUSTER DECAY.

MANY CLUSTERS

CONSIDER MANY CLUSTERS LOCATED AT THE POINTS Y_m
DISTRIBUTED AROUND THE POINTS $\bar{Y}_m = m\Delta$,
 $m = -N, \dots, N$ ACCORDING TO

$$F_m(Y_m) = (Y_0\sqrt{\pi})^{-1} e^{-(Y_m - \bar{Y}_m)^2 / Y_0^2} \quad (26)$$

IF THE CLUSTERS ARE INDEPENDENT, THE GENERATING
FUNCTION IN THE BIN δ IS

$$\Phi(z, \delta) = \prod_{m=-N}^N \Phi_m(y, \delta) \quad (27)$$

WITH $\Phi_m(z, \delta) = \int_{-\infty}^{\infty} dY F_m(Y) e^{-\bar{n}g(\delta, Y)(z-1)}$.

$$\Phi(z, \delta) = \int \left[\prod_{m=-N}^N dY_m F(Y_m) \right] e^{(z-1) \sum_{m=-N}^N \bar{n}g(\delta, Y_m)} \quad (28)$$

FACTORIAL MOMENTS

FROM THE GENERATING FUNCTION ONE CAN EVALUATE THE FACTORIAL MOMENTS

$$F_k(\delta) = \frac{d^k \phi(z, \delta)}{dz^k} [z = 1]; \quad f_k \equiv F_k / F_1^k; \quad F_1 = \langle n \rangle \quad (29)$$

FOR NEGATIVE BINOMIAL $\phi(z) = [1 + \langle n \rangle (1 - z) / \kappa]^{-\kappa}$

$$f_2 = 1 + 1/\kappa; \quad f_3 = (1 + 1/\kappa)(1 + 2/\kappa) \quad (30)$$

I.E. $1/\kappa_2 = f_2 - 1$ AND $1/\kappa_3 = [\sqrt{1 + 8f_3} - 3]/4$.

THE DIFFERENCE BETWEEN $1/\kappa_2$ AND $1/\kappa_3$ MEASURES DEVIATION FROM THE NEGATIVE BINOMIAL.

I HAVE EVALUATED NUMERICALLY f_2 AND f_3 OF THE DISTRIBUTION IN THE BIN δ FOR VARYING BIN SIZE AND VARIOUS VALUES OF Δ (AVERAGE DISTANCE BETWEEN CLUSTERS) AND Y_0 (THE WIDTH OF THE CLUSTER POSITION AROUND ITS AVERAGE).

NEGATIVE BINOMIAL ?

$$\Delta = Y_0 = 2, 1, 0.5$$

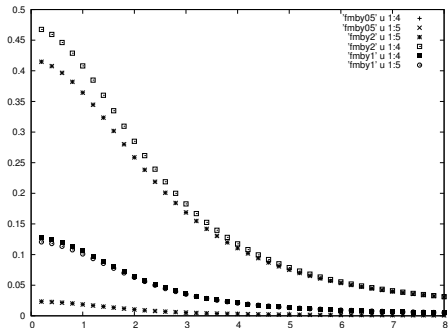


Figure: The parameter $1/k$ determined from f_2 and from f_3 plotted versus the length of the rapidity bin. The three groups of curves correspond to $\Delta = 2$, $\Delta = 1$ and $\Delta = 0.5$ (Δ is the distance between clusters). The distribution of clusters is $\sim e^{-(Y-Y_i)^2/Y_0^2}$ with $Y_0 = \Delta$.

NEGATIVE BINOMIAL ?

$$\Delta = 1 \quad Y_0 = 2, 1, 0.5$$

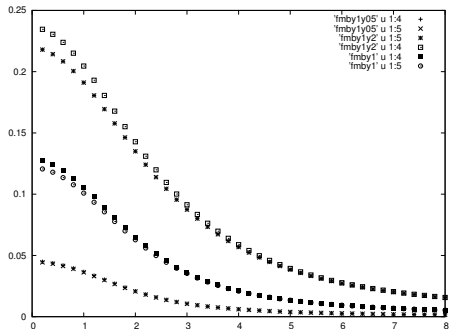


Figure: The parameter $1/k$ determined from f_2 and from f_3 plotted versus the length of the rapidity bin, for $\Delta = 1$. The three groups of curves correspond to $Y_0 = 2$, $Y_0 = 1$ and $Y_0 = 0.5$

NEGATIVE BINOMIAL ?

$$Y_0 = 1 \quad \Delta = 2, 1, 0.5$$

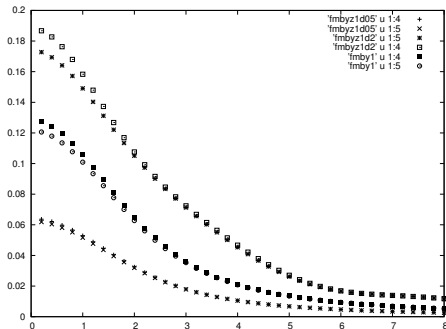


Figure: The parameter $1/k$ determined from f_2 and from f_3 plotted versus the length of the rapidity bin, for $Y_0 = 1$. The three groups of curves correspond to $\Delta = 2$, $\Delta = 1$ and $\Delta = 0.5$