# High temperature topological susceptibility in QCD: results and open problems

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Gauge topology: from lattice to colliders
Trento, 7 November - 11 November, 2016

# Outline

1. Theoretical expectations
2. Lattice results
   - without light fermions
   - with light fermions
3. Problems (solved and to be solved)

# The general form of $F(\theta, T)$

$$F(\theta, T) = -\frac{1}{V_4} \log \int [\mathscr{D}A][\mathscr{D}\bar{\psi}][\mathscr{D}\psi] \exp\left( -\int_0^{1/T} \mathrm{d}t \int \mathrm{d}^3x \, \mathcal{L}_\theta^E \right)$$

Assuming analiticity at $\theta = 0$ the free energy density can be written as:

$$F(\theta, T) - F(0, T) = \frac{1}{2}\chi(T)\theta^2 \left[ 1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \cdots \right] ,$$

with

$$\chi = \frac{1}{V_4}\langle Q^2 \rangle_0 \qquad b_2 = -\frac{\langle Q^4 \rangle_0 - 3\langle Q^2 \rangle_0^2}{12\langle Q^2 \rangle_0}$$

$$b_4 = \frac{\langle Q^6 \rangle_0 - 15\langle Q^2 \rangle_0 \langle Q^4 \rangle_0 + 30\langle Q^2 \rangle_0^3}{360\langle Q^2 \rangle_0} ,$$

where $\langle \quad \rangle_0$ denotes the averages at $\theta = 0$.

Coefficients $b_{2n}$ parametrize deviations of the distribution of topological charge from a Gaussian in the theory at $\theta = 0$.

# Large $N$ and $\chi PT$

## Large $N$

The scaling variable to keep fixed is $\bar{\theta} \equiv \theta/N$ and one gets (Witten 1980)

$$\chi = \bar{\chi} + \cdots$$
$$b_{2n} = \bar{b}_{2n}/N^{2n} + \cdots$$

## Chiral perturbation theory

LO at $T = 0$ (Di Vecchia, Veneziano 1980)

$$E_0(\theta) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2 \frac{\theta}{2}}$$

NLO (G. Grilli di Cortona, E. Hardy, J. Pardo Vega, G. Villadoro 1511.02867)

| | | |
|---|---|---|
| $z \equiv m_u/m_d = 0.48(3)$ | $\chi^{1/4} = 75.5(5)\,\mathrm{MeV}$ | $b_2 = -0.029(2)$ |
| $z = 1$ | $\chi^{1/4} = 77.8(4)\,\mathrm{MeV}$ | $b_2 = -0.022(1)$ |

# Dilute Instanton Gas Approximation (1)

Hypothesis: the dynamic of the system is dominated by weakly interacting objects of topological charge $\pm 1$.

This is surely true in the weak coupling approximation ($T \gg \Lambda_{QCD}$).

In the DIGA approximation we thus have (Gross, Pisarski, Yaffe 1981)

$$Z_\theta = \mathrm{Tr} e^{-H_\theta/T} \approx \sum \frac{1}{n_+! n_-!} (V_4 D)^{n_+ + n_-} e^{-S_0(n_+ + n_-) + i\theta(n_+ - n_-)}$$
$$= \exp\left[ 2 V_4 D e^{-S_0} \cos\theta \right]$$

where $1/D$ is a typical $4-$volume, that in perturbation theory is related to the functional determinants of the fields in the instanton background.
Using DIGA without perturbation theory we thus have:

$$F(\theta, T) - F(0, T) \approx \chi(T)(1 - \cos\theta)$$

# Dilute Instanton Gas Approximation (2)

Using only the DIGA hypothesis we have informations on the explicit values of $b_{2n}(T)$ but not on $\chi(T)$:

$$b_2 = -\frac{1}{12} \quad b_4 = \frac{1}{360} \quad b_{2n} = (-1)^n \frac{2}{(2n+2)!}$$

Using also perturbation theory $S_0 = \frac{8\pi^2}{g^2(T)} \simeq (\frac{11}{3}N - \frac{2}{3}N_f)\log(T/\Lambda)$ and close to the chiral limit $D \propto T^4(m/T)^{N_f}$, so that

$$\chi(T) \propto m^{N_f}\, T^{4-\frac{11}{3}N-\frac{1}{3}N_f}$$

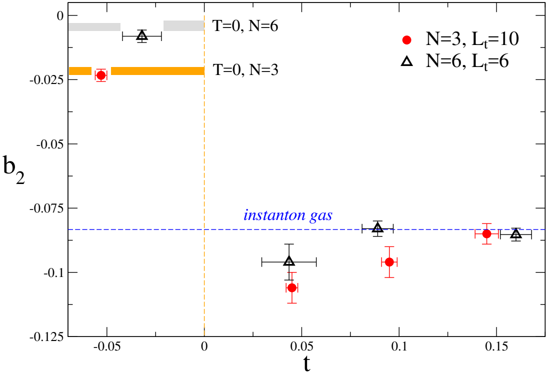
(Gross, Pisarski, Yaffe 1981)

# $SU(N)$ theory without matter

C. Bonati, M. D'Elia, H. Panagopoulos, E. Vicari 1301.7640

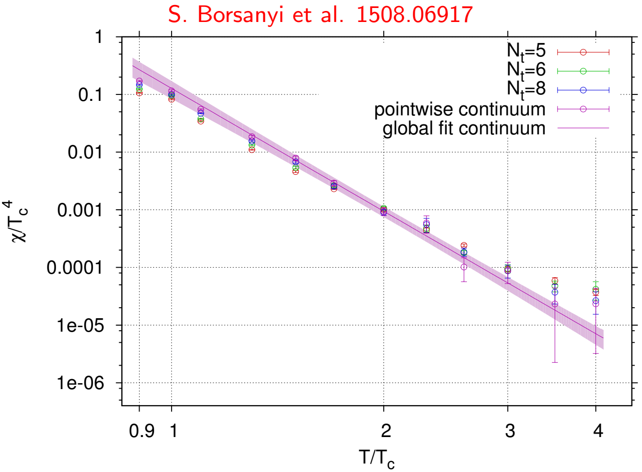
(and 1512.01544 and 1607.06360)

$$t = (T - T_c)/T_c$$




- large-$N$ scaling for $T < T_c$, $b_2$ independent of $N$ for $T > T_c$
- DIGA values ($b_2 = -1/12$, $b_4 = 1/360$) reached for $T \gtrsim 1.1 T_c$

# $SU(3)$ theory for $T > T_c$

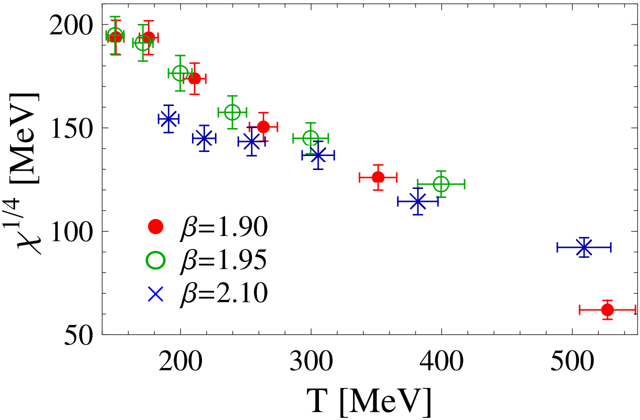



$\chi(T) \propto 1/T^b$, where $b = 7.1(4)(2)$.

# $SU(3)$ with light fermions: literature

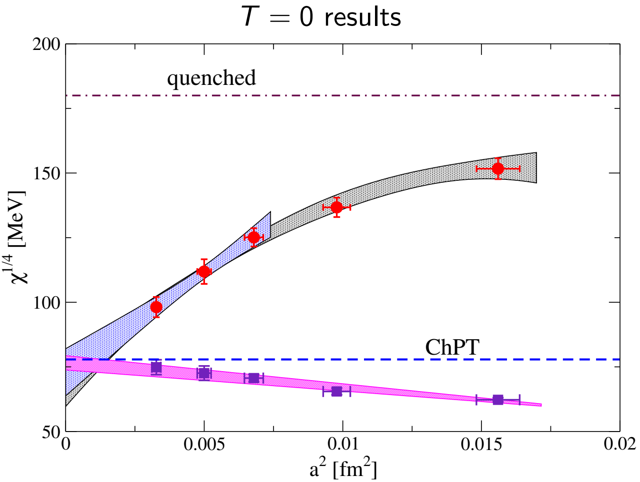
- A. Trunin, F. Burger, E. M. Ilgenfritz, M. P. Lombardo and M. Müller-Preussker, J. Phys. Conf. Ser. **668**, no. 1, 012123 (2016) [arXiv:1510.02265 [hep-lat]].
  2+1+1 TM Wilson, $m_\pi \simeq 370\,\mathrm{MeV}$
- C. Bonati, M. D'Elia, M. Mariti, G. Martinelli, M. Mesiti, F. Negro, F. Sanfilippo and G. Villadoro, JHEP **1603**, 155 (2016) [arXiv:1512.06746 [hep-lat]].
  2+1 stout staggered fermions at the physical point
- P. Petreczky, H. P. Schadler and S. Sharma, Phys. Lett. B **762**, 498 (2016) [arXiv:1606.03145 [hep-lat]].
  2+1 HISQ fermions, $m_\pi \simeq 160\,\mathrm{MeV}$
- Sz. Borsanyi, Z. Fodor, K. H. Kampert, S. D. Katz, T. Kawanai, T. G. Kovacs, S. W. Mages, A. Pasztor, F. Pittler, J. Redondo, A. Ringwald, K. K. Szabo, Nature **539** no.7627, 69 (2016) [arXiv:1606.07494 [hep-lat]].
  2+1+1 stout staggered fermions and 2+1 overlap fermions at the physical point

# Results from 1510.02265 (Trunin et al.)




"The topological susceptibility exhibits a surprisingly slow decrease at high temperature."

# Results from 1512.06746 (Bonati et al.)




violet data: $a\chi^{1/4}(a)\frac{m_\pi^{\mathrm{phys}}}{am_{ngb}(a)}$

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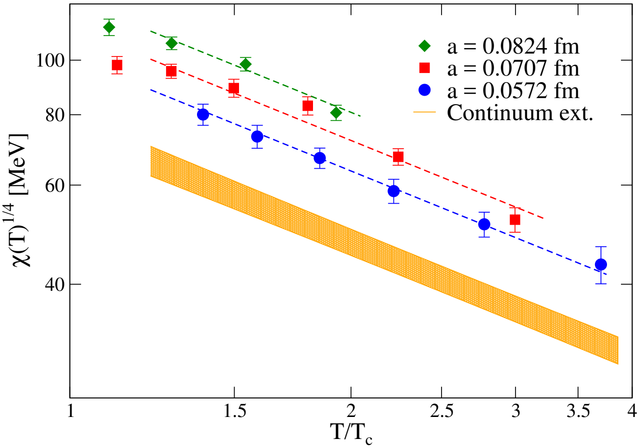
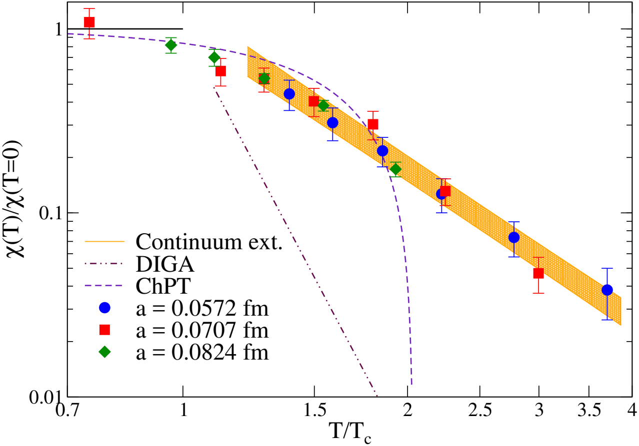



fit with $\chi(T)^{1/4} = A_0(1 + A_1 a^2)(T/T_c)^{A_2}$

strong lattice artefacts and $4A_2 \simeq -3$.

# Results from 1512.06746 (Bonati et al.)




All the (visible) lattice artefacts disappears in the ratio,
still $\chi(T) \sim T^{-3}$

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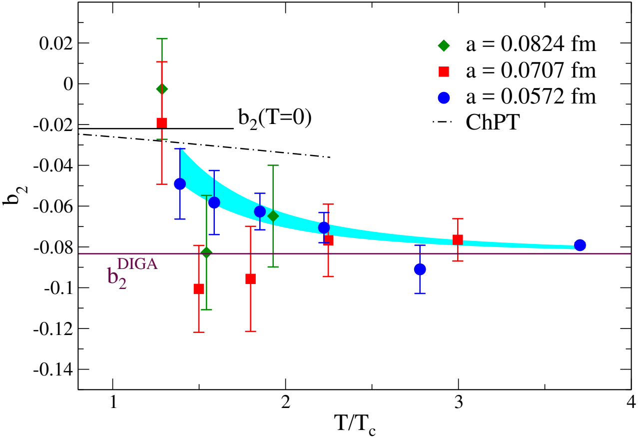
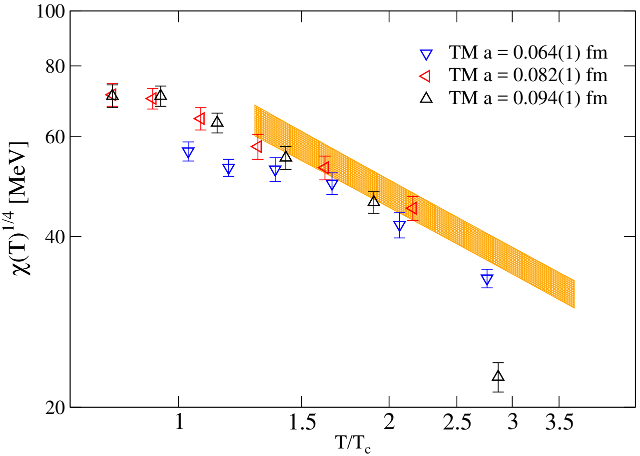


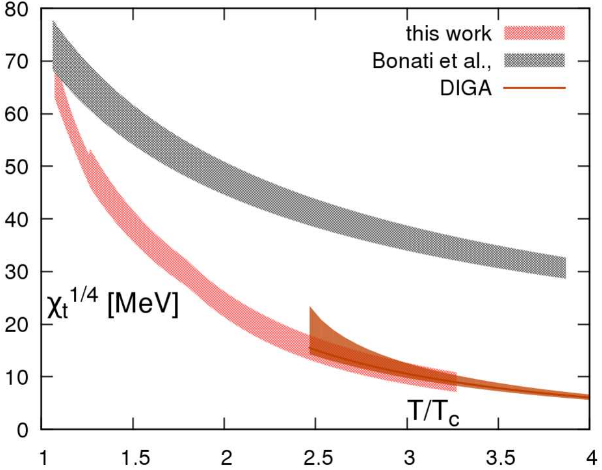

Convergence to the asymptotic value slower than in pure gauge theory, convergence from above.

# Results from 1512.06746 (Bonati et al.)



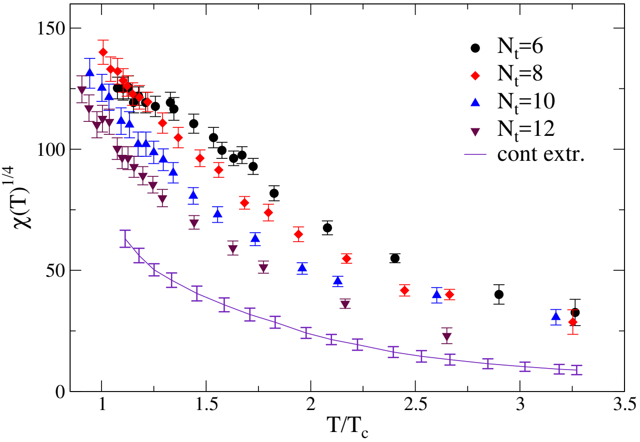

Comparison with the result by Trunin et al.,
$\chi(T)$ rescaled by $m_q^2 \sim m_\pi^4$.

# Results from 1606.03145 (Petreczky et al.)



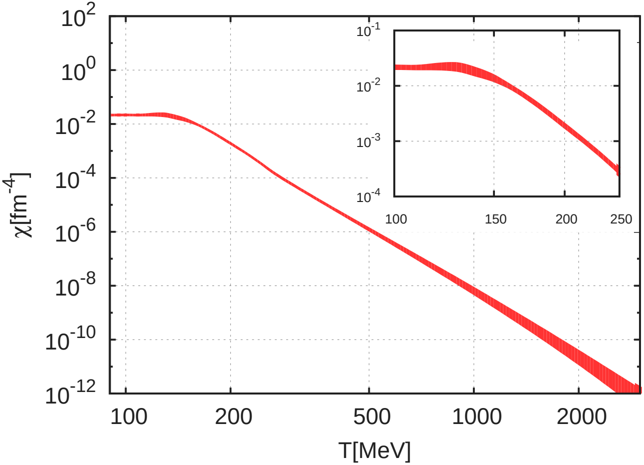

Result close to the DIGA value.

# Results from 1606.03145 (Petreczky et al.)




But again large lattice artefacts.

# Results from 1606.07494 (Borsanyi et al.)




Result compatible with DIGA

# Problem 1: definition of topology on the lattice

The topological charge is well defined only for smooth enough gauge configuration, so its definition on the lattice require some care.

Several methods have been devised during the years to study topology on the lattice:

- Field theoretical methods (perturbative/nonperturbative computation of the finite renormalization constants)
- Fermionic methods (using the lattice index theorem for Ginsparg-Wilson fermions)
- Smoothing methods

All these methods have advantages and drawbacks, nevertheless they have been proven to give compatible results for the physical observables (see e.g. Panagopoulos, Vicari 0803.1593, Bonati, D'Elia 1401.2441).

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SOLVED

# Problem 2: lack of self-averaging for $b_{2n}$

Consider e.g.

$$b_2 = -\frac{\langle Q^4\rangle_0 - 3\langle Q^2\rangle_0^2}{12\langle Q^2\rangle_0}$$

In the thermodynamic limit the probability distribution of $Q$ is dominated by gaussian fluctuation of typical size $\delta Q \sim \sqrt{\chi V_4}$, thus the mean value of the numerator grows $\sim V_4$, while its error $\sim \chi V_4^2$.

Lack of self-averaging

Standard solution: do not study fluctuation observables "at zero external field", study instead the response to an external field (Milchev, Binder, Heermann 1986).

In this case "external field" = nonvanishing $\theta$. To avoid the sign problem use $\theta = i\theta_I$, with $\theta_I \in \mathbb{R}$ (Vicari, Panagopoulos 1109.6815). Never tried with fermions up to now, possible problems related to additive renormalization of $\theta$?

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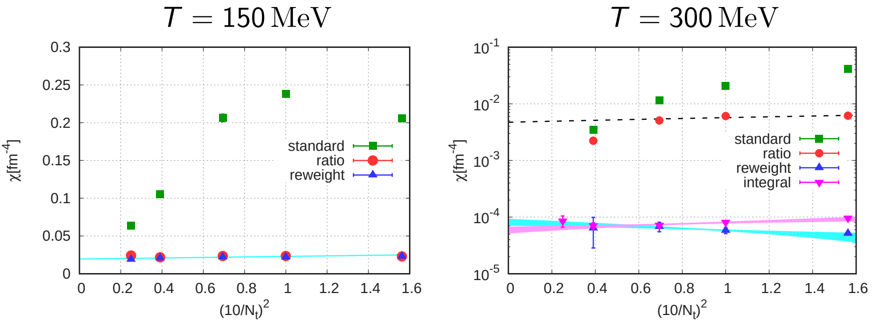
PARTIALLY SOLVED

# Problem 3: lattice chiral symmetry breaking

Apart from some computationally very expensive choice (like overlap), the discretized fermions break chiral symmetry in some way.

$\chi(T)$ is extremely sensitive to the quark masses as the chiral symmetry is approached: large lattice artefacts.

First try (Bonati et al. 1512.06746): use $\chi(T,a)/\chi(T=0,a)$.




from Borsanyi et al. 1606.07494

# Problem 3: lattice chiral symmetry breaking (2)

Proposal from Borsanyi et al. 1606.07494: eigenvalue reweighting.

For staggered fermions the weight factor should be

$$w[U] = \prod_f \prod_{\substack{\text{would be} \\ \text{zero modes}}} \left( \frac{2m_f}{i\lambda_n[U] + 2m_f} \right)^{n_f/4}$$

This procedure drastically reduces the dependence of $\chi(T)$ on the lattice spacing and it has been shown to be quite insensitive to the particular definition of “would be zero mode” adopted.

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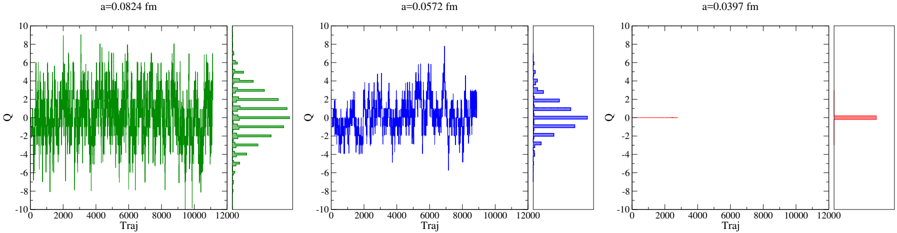
MAYBE SOLVED

# Problem 4: freezing of the topological charge

As the continuum limit is approached it gets increasingly difficult to correctly sample the various topological sectors.

exponential critical slowing-down




from Bonati et al. 1512.06746

Note: this is an algorithmic problem, not a physical problem.

# Problem 4: freezing of the topological charge (2)

Various proposal to solve or alleviate this problem:

1. Algorithms to remove the barriers: boundary conditions change, metadynamic, . . .
2. Algorithms to get results also when $Q$ is frozen: large volume expansion, slab method, "thermodynamic integration", . . .

By now no "official winner" in the fight between the algorithms of type 1).

For algorithms of type 2): beware of the correlations between $Q$ and other observables (see e.g. D'Elia, Negro 1306.2919).

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WORK IN PROGRESS

# Problem 5: thermodynamic limit

As $T$ grows $\chi(T)$ approaches zero and this has two practical consequences in simulation (obviously at finite volume)

1. effective "freezing" at $Q = 0$
2. possibly large finite volume effects

Note: this is a physical problem, not just an algorithmic one.

A solution to problem 1) is provided by "thermodynamic integration"
Frison et al. 1606.07175, Borsanyi et al. 1606.07494.
For $\chi V_4 \ll 1$ we have

$$\chi \simeq \frac{1}{V_4}\frac{Z_{-1}+Z_1}{Z_0} = \frac{2}{V_4}\int_0^\beta \frac{\mathrm{d}}{\mathrm{d}\bar{\beta}}\frac{Z_1}{Z_0}\mathrm{d}\bar{\beta} = \frac{2}{V_4}\int_0^\beta \left(\langle S\rangle_{\bar{\beta},1} - \langle S\rangle_{\bar{\beta},0}\right)\mathrm{d}\bar{\beta}$$

Is this value the correct thermodynamic value of $\chi$? Only if instantons are weakly interacting, but to check if this is the case we need to study the regime $\chi V_4 \gg 1$.

# Problem 5: thermodynamic limit (2)

An indication that we are far from the thermodynamic limit is the following (Azcoiti 1609.01230).

In Frison et al. 1606.07175, Borsanyi et al. 1606.07494 is numerically seen that

$$\langle S \rangle_Q - \langle S \rangle_{Q=0} \propto Q$$

and this value is independent of the volume, while on general grounds one has that in the thermodynamic limit

$$\langle S \rangle_Q - \langle S \rangle_{Q=0} \propto Q^2 / V_4$$

(Brower et al. 0302005, Aoki et al. 0707.0396, Dromard et al. 1404.0247).

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STILL A BIG CHALLENGE

# Conclusions

For theories with no fermions everithing is by now well established:

- at deconfinement the $\theta$ dependence switches from the large-$N$ behaviour in the low-$T$ phase to an "instanton gas" behaviour in the high-$T$ phase
- in the deconfined phase $\chi(T) \propto T^{-7}$ basically starting from $T_c$ (explanations alternative to DIGA+PT?)

For theories with light fermions

- we are able to reproduce the $\chi PT$ result at $T = 0$
- as in the case of pure gauge theories $\chi(T) \simeq \chi(0)$ for $T \lesssim T_c$
- the case $T > T_c$ still require further study, the main obstructions being the slowing down of the topological modes and the finite size effects.

Thank you for your attention!

# Backup slides with something more

# A simple example of instanton interaction

Let us consider the case of a gas of hard-core instantons and anti-instantons: in this case

$$Z(\theta) = \sum_{n_+ + n_- \leq DV_4} \frac{1}{n_+! n_-!} \frac{(V_4 D)!}{(V_4 D - n_+ - n_-)!} e^{-S_0(n_+ + n_-) + i\theta(n_+ - n_-)} =$$

$$= \left(1 + 2e^{-S_0} \cos\theta\right)^{V_4 D}$$

from which

$$\chi_{HC} = \frac{2De^{-S_0}}{1 + 2e^{-S_0}}$$

On the other hand when $\chi V_4 \ll 1$ we have

$$\chi_{\mathrm{small}} = 2Z_1/(Z_0 V_4) = 2De^{-S_0} = \chi_{DIGA} \neq \chi_{HC}$$