# Magnetic backgrounds, topology of space-time, and the holonomy

**Massimo D'Elia**

**University of Pisa & INFN**

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## In this workshop, we discussed topology in a wide sense:

- Topology of semi-classical solutions of gauge theories (instantons, monopoles, vortices, dyons). Their role in confinement, deconfinement, $\theta$-dependence

- $\theta$-dependence in QCD and axions

- Topology in external backgrounds, chiral properties of gauge theories

- Topology of space-time (compactification), boundary conditions and phases of gauge theories

  Ideas from topology exported to biology

- I would like to spend a few more words on an entanglement between topology of space-time and external background fields
  (ongoing work with Marco Mariti)

## QUESTION

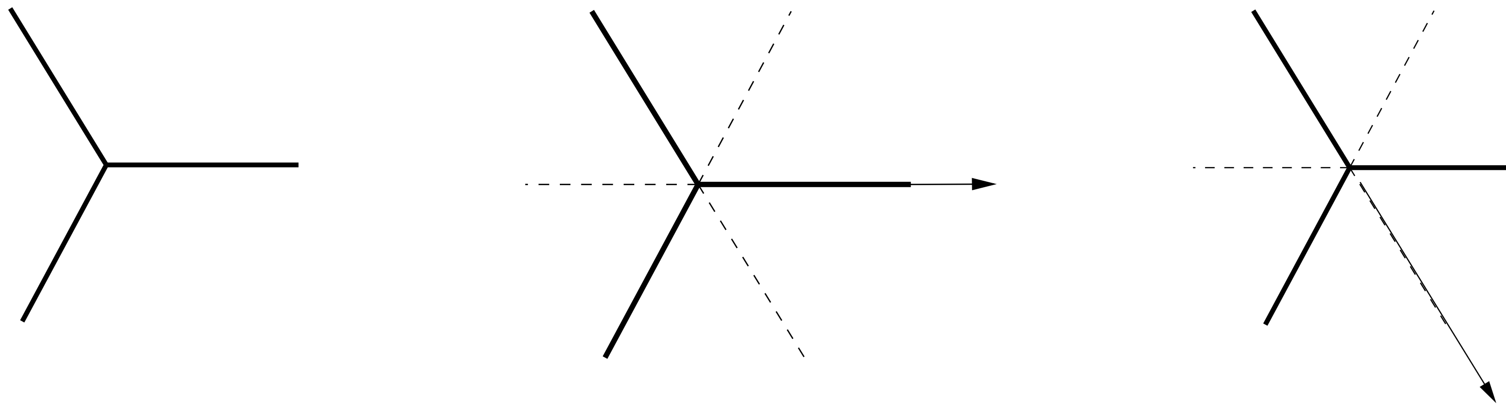
**what happens if we compactify one dimension in the presence of an electromagnetic background $F_{\mu\nu}$?**

## In brief:

- **the electromagnetic field couples to the holonomy in the compactified direction (like an imaginary chemical potential does)**
- **center symmetry is explicitly broken by the presence of dynamical fermions, however the selected center sector, because of the coupling to the e.m. field, becomes space dependent**
- **For small compactifications, due to the perturbative holonomy potential, local minima develop in each center sector, so that changing center sector gives rise to interfaces, which cost energy.**

  **Hence, a complicated pattern of phase transitions can arise at which interfaces are created or destroyed, depending on the balance between compactification radius and background field strength.**

Let us remember the interesting things happening in the $T$ imaginary chemical potential plane, in connection with the realization of center symmetry. **Imaginary $\mu = i\mu_I$ rotates fermion temporal b.c. by $\mu_I/T$**

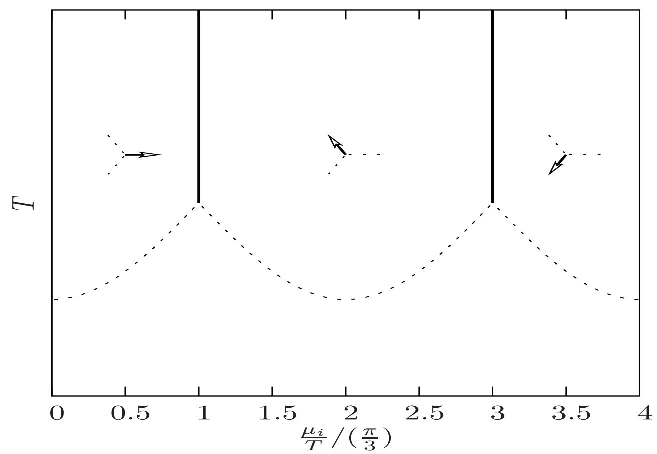


**Pure gauge theory**
**Exact $Z_3$ center symmetry**
**Spontaneously broken at $T > T_c$**

**Full QCD, $\mu = 0$**
**Determinant couples to holonomy**
**Its trace is always real**

**Full QCD, $\theta_I \equiv \mu_I/T \neq 0$**
**Determinant couples to holonomy * $e^{i\theta_I}$**
**Minimum for the holonomy rotates with $\theta_I$**

**In the high $T$ region, the absolute minimum (i.e. the preferred center sector) is decided by $\theta_I$. That leads to the appearance of Roberge-Weiss transitions as a function of $\theta_I$ (one every $2\pi/N_c$ interval)**



**A general $U(1)$ background field, coupled to the electric charge of quark, will do a similar thing. The covariant derivative is modified as follows**

$$D_\nu = \partial_\nu + i\, g A_\nu^a T^a + i\, q a_\nu \tag{1}$$

**where $T^a$ are the $SU(3)$ generators and $q$ is the coupling to the external $U(1)$ field $a_\mu$. For simplicity we consider quarks degenerate in mass and electric charge.**

**If a dimension is compactified, the coupling of dynamical fermions to the holonomy involves the external field directly, i.e. the coupling is to**

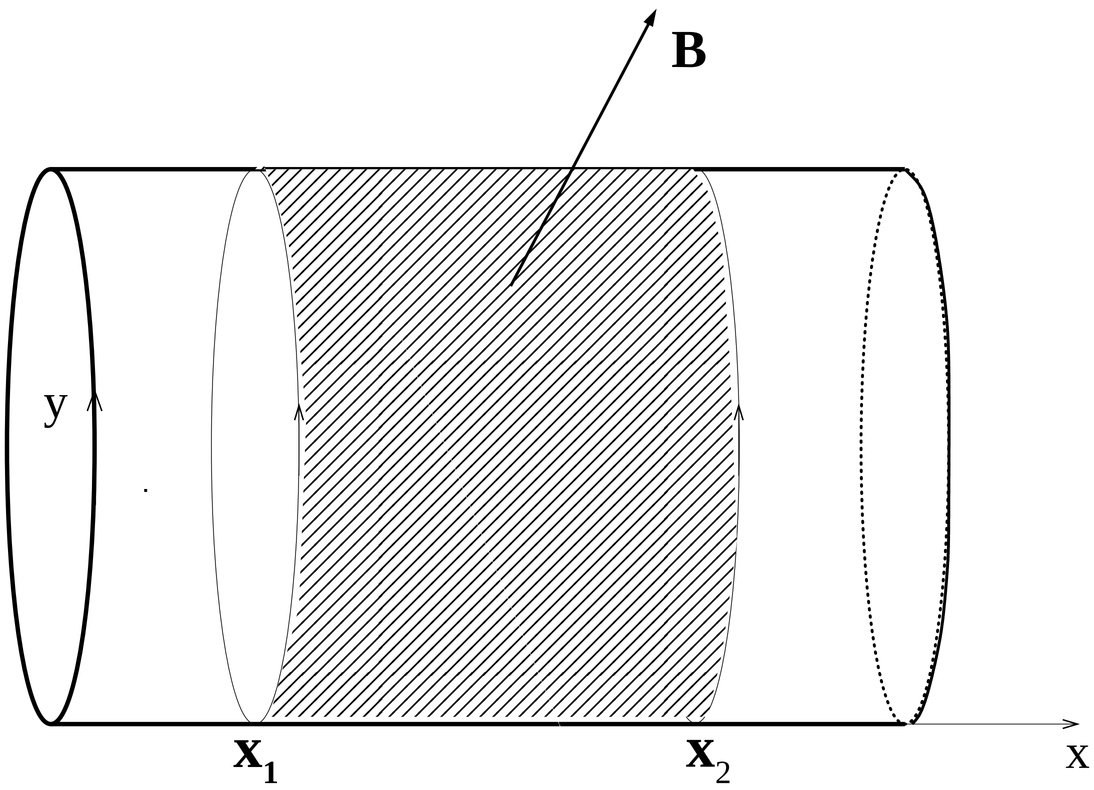
$$\mathrm{Tr}\exp\left(\int dx_\mu i(gA_\mu + qa_\mu)\right) = L e^{i\phi(\vec{x})}\,.$$

**where $\phi(\vec{x})$ is a phase factor which, contrary to the case of an imaginary chemical potential, can depend on the coordinates $\vec{x}$ orthogonal to the compactified direction**

**That will give rise to a coupling to holonomy which changes from point to point, that will give rise to different orientations of the holonomy and to the rise of an interesting phenomenology.**

## The specific example we will consider:

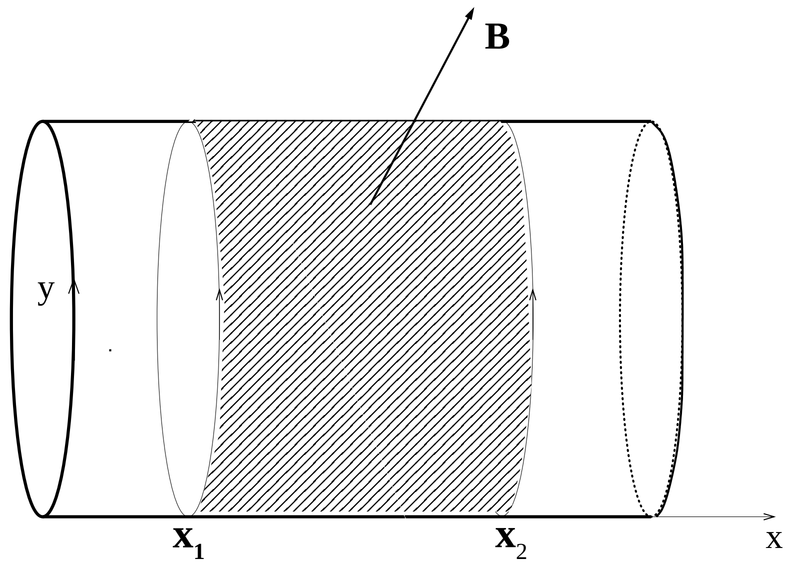
### A cilinder with a magnetic background orthogonal to its surface




one can view this as a spatial compactified dimension with a magnetic field. However, because of the spatially varying boundary conditions, thermal or non-thermal is completely irrelevant, so a thermal gauge theory with an imaginary electric field is perfectly equivalent.

The value of a single phase factor is not relevant and depends on the $U(1)$ gauge choice.

However phase differences are meaningful and gauge invariant




$$e^{i(\phi(\mathbf{x_2})-\phi(\mathbf{x_1}))} = \exp\left(iq\int dy(a_y(\mathbf{x_2},y) - a_y(\mathbf{x_2},y))\right) = e^{iq\Phi_\mathbf{B}}$$

where $\Phi_\mathrm{B}$ is the total magnetic field flux going through the shadowed surface.

The value of such a flux is, for any given distribution of magnetic field, a property of the points $\mathrm{x}_1$ and $\mathrm{x}_2$ only (also in higher dimensions)

Therefore, we expect that $N_c q\Phi_B/(2\pi)$ different center sectors should be explored.

For small compactification, when the holonomy develops local minima, that would imply the formation of $N_c q\Phi_B/(2\pi)$ center interfaces between $\vec{x}_1$ and $\vec{x}_2$.

# Does the “lattice” of center domains and interfaces really form?

Interfaces have an energy cost, could it be more convenient to stay (locally) in the wrong sector without forming any interface?

This is an issue of energy balance. We will make an estimate of such energy balance under some simplified assumption:

- The magnetic background is uniform and constant
- The compactification radius is so small that perturbative estimates for the interface tension (or for the false vacuum energy) hold true

# Uniform background field and exact center-translational symmetry

When the background field is uniform, center symmetry, which is explicitly broken by the presence of dynamical fermions, is recovered in a different form:

One can rotate temporal boundary conditions for gauge fields by $-2\pi/N_c$, and perform at the same time a translation along $x$ by $2\pi/(qBL_cN_c)$, where $L_c$ is the size of the compactified direction the theory will be mapped onto itself

This discrete symmetry can be:

1. Realized exactly:

   After each translation by $2\pi/(qBL_cN_c)$ the holonomy rotates by $-2\pi/N_c$, with more or less sharp interfaces (more and more sharp as $L_c$ decreases)

2. Spontaneously broken:

   The holonomy fails to rotate as you translate, because interfaces cost too much and it is more convenient to stay in the false vacuum somewhere.

# ENERGY BALANCE - I

We need to compute the balance between

- **the energy spent in creating center interfaces**

  this is a function of the interface tension and of the density of interfaces, which depends on the magnetic field strength

- **the energy spent in keeping the holonomy in a locally wrong vacuum**

  This is a function of the holonomy effective potential.

**Let $L \equiv x_2 - x_1$. We consider two extreme situations:**

- **All center domains are actually formed:** the number of interfaces, $N_{int}$, is given by all the different center sectors spanned by the local phase between $x_1$ and $x_2$:

$$N_{int} = q\Phi_B/(2\pi/N_c) = qBLL_cN_c/2\pi \; , \tag{2}$$

- **the holonomy stays in the same center sector everywhere, no interface is formed**

  one must keep the holonomy in the wrong center sector for a fraction $(N_c - 1)/N_c$ of the region between $x_1$ and $x_2$.

# ENERGY BALANCE - II

In the limit of asymptotically small $L_c$, we can recover perturbative results obtained in thermal field theory, where $L_c = 1/T$

- The interface tension (energy per unit interface area) is proportional to $L_c^{-3}\,\log(1/L_c)$
- The energy density to keep the holonomy in the wrong vacuum is prop. to $L_c^{-4}$.

Apart from a common factor related to the integration in the non-compactified directions

- the energy spent to create all possible interfaces between $x_1$ and $x_2$ is prop. to

$$qBLL_c^{-2}\log(1/L_c)$$

- the energy spent to maintain the holonomy in the same center sector, without creating any interface, is proportional to

$$LL_c^{-4}$$

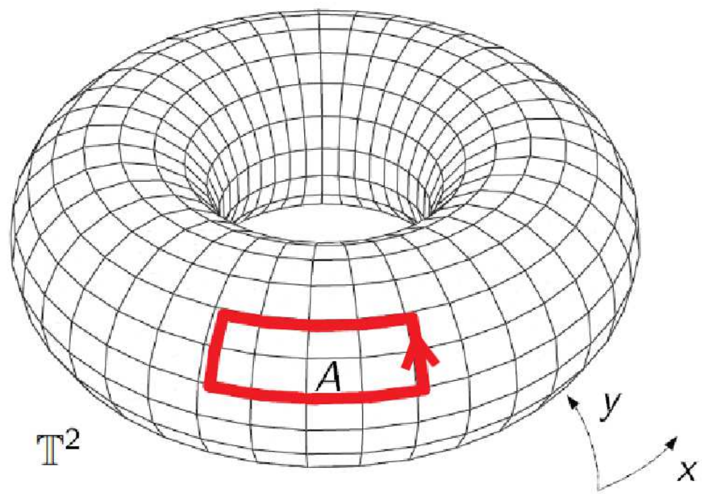
The first situation is surely favored, at fixed magnetic field, for small enough $L_c$ and, at fixed $L_c$, for small enough $B$.

Increasing $L_c$ or $B$ can make instead the second situation more favorable.

# Some numerical tests from lattice simulations

We have performed lattice simulations with rooted staggered fermions

- 2 flavors, degenerate both in mass and charge, pion mass quite large (800 MeV)
- magnetic field along $\hat{z}$, $y$ is the short compactified direction, $x$ is compactified as well, but of length $L_x = L \gg L_c$
- Since we are on a torus, magnetic field (actually, magnetic flux) is quantized.
- We will consider simulations at fixed $B$ and see what happens when decreasing $L_c$, or at fixed $L_c$ and see what happens when changing $B$

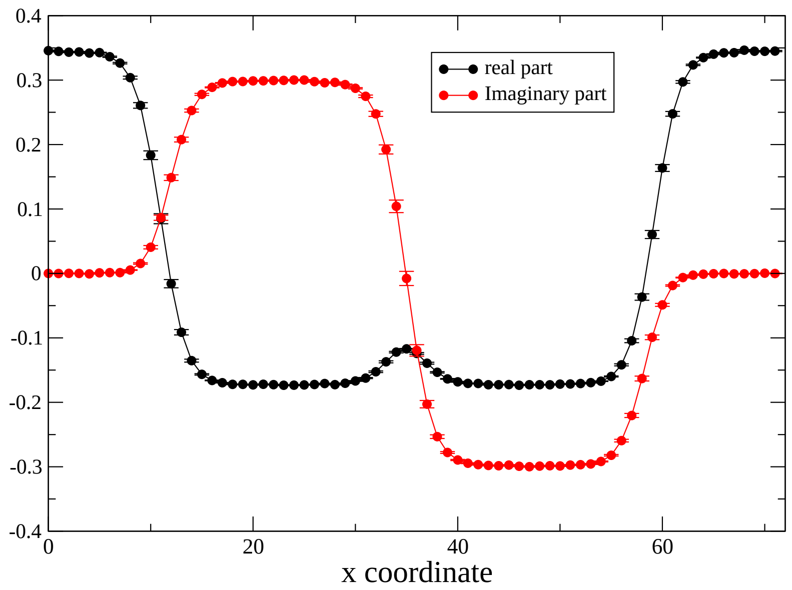


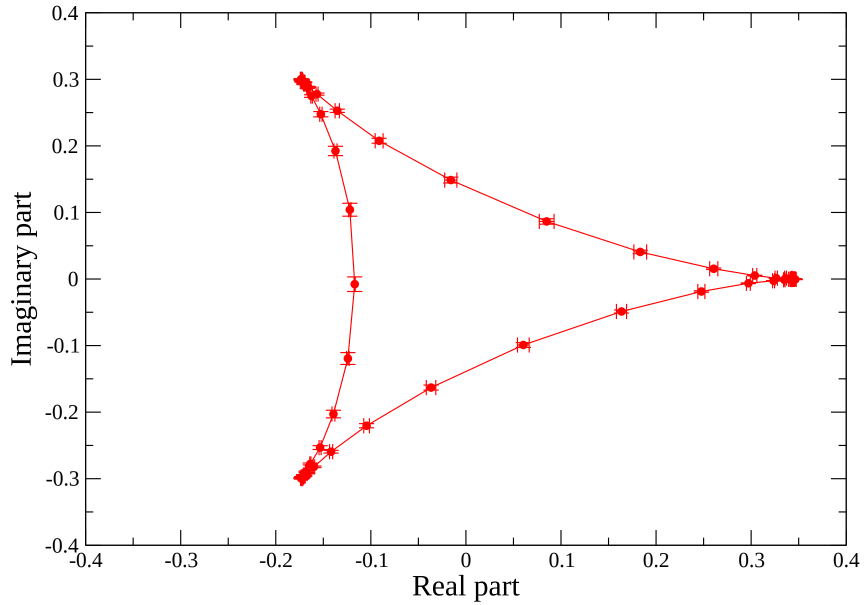
Simulations are done at fixed bare parameters ($\beta = 6.2$, $am = 0.01$)

$L_z = L_t = 24$ for all simulations

$L_y$ is the compactified direction $L_c$

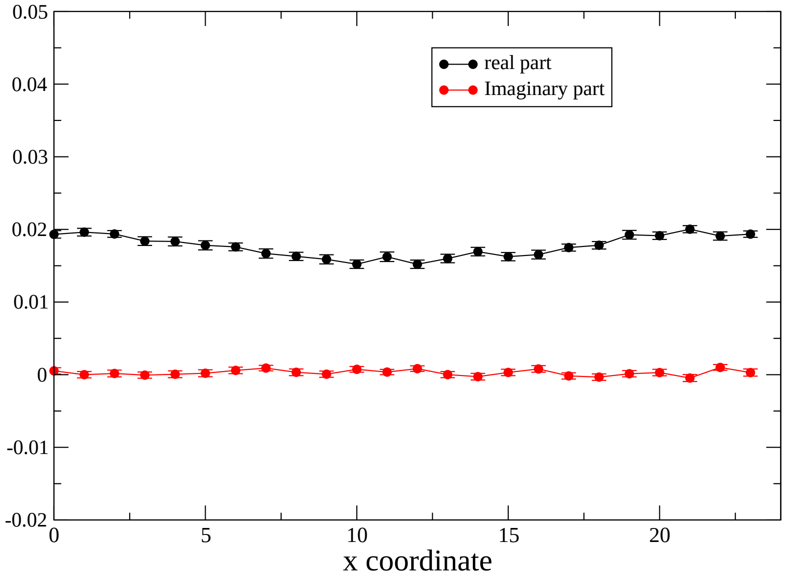
$L_x$ is changed so as to tune $qB = 2\pi b/(L_x L_y a^2)$ where $b$ is an integer.

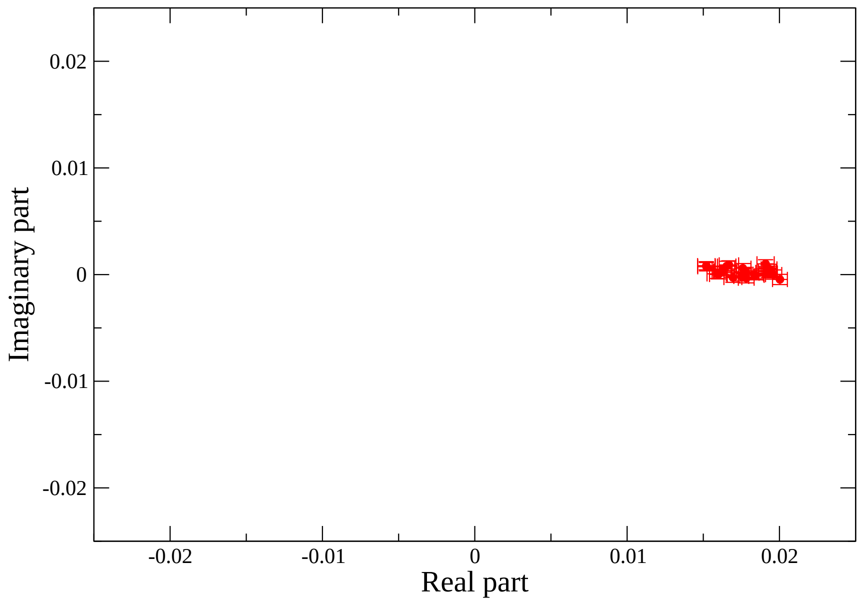



We start by showing the behavior of the local values taken by the holonomy (at fixed $x$ coordinate) for the case: $b = 1$, $L_c = 4$, $L_x = 72$

For the same values, at $B = 0$ the system is in the deconfined phase, with a non-zero, real holonomy. At $B \neq 0$ the formation of center domains and center interfaces is clearly visible

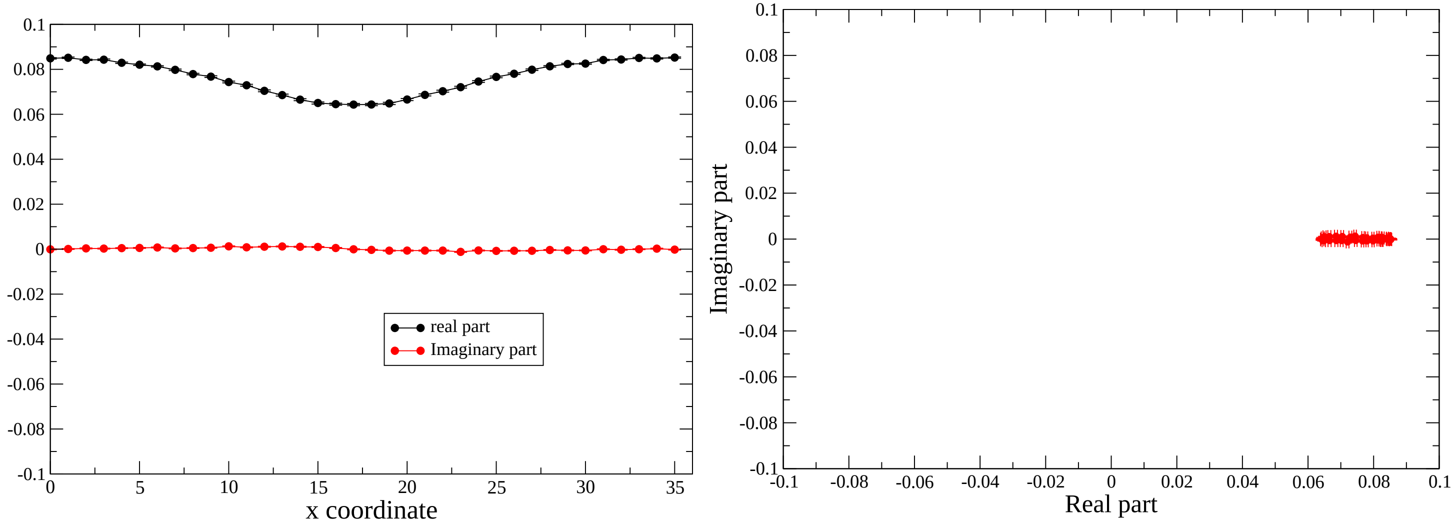



**Next we enlarging $L_c$ at fixed $B$:**

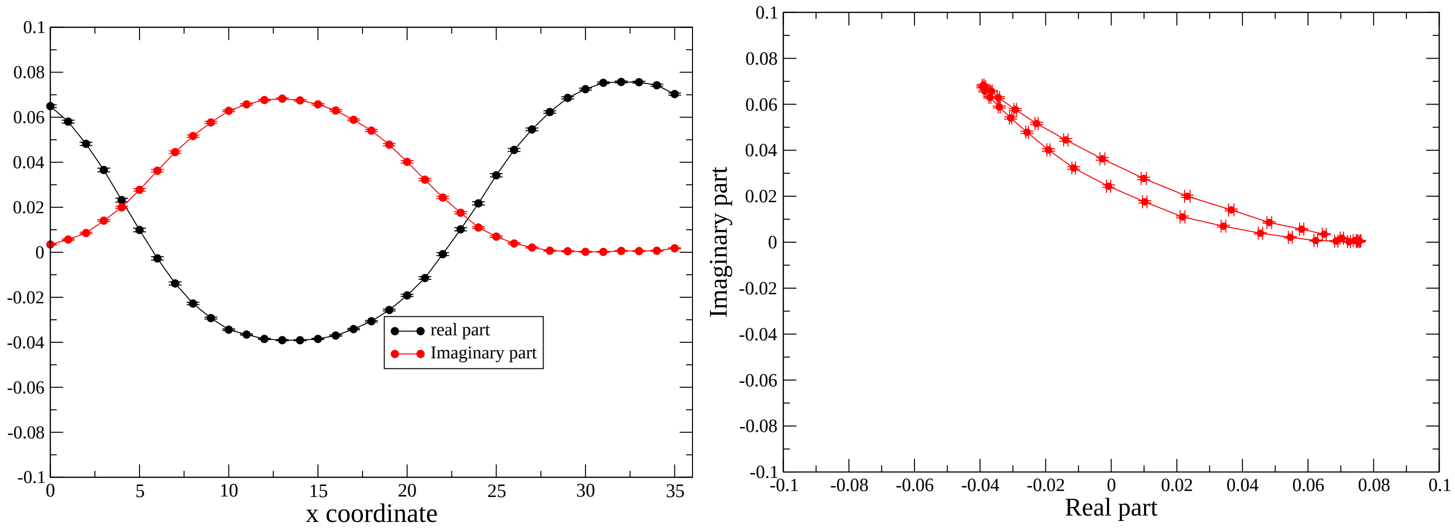
$b = 1$, $L_c = 12$, $L_x = 24$

**The system is found in a different phase, the center-translational symmetry is broken spontaneously, a given center sector is selected throughout the lattice**



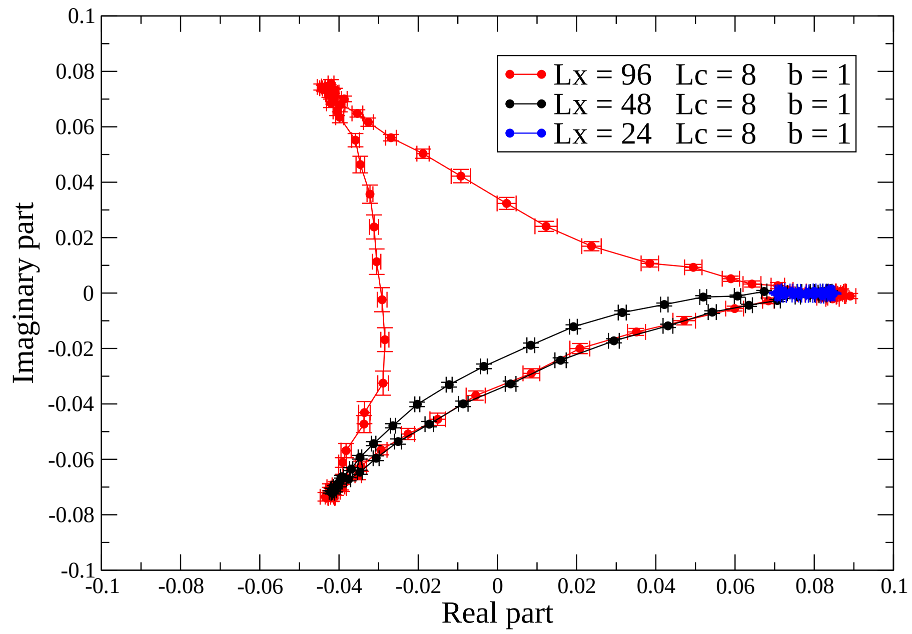

**We can even have different phases:** $b = 1$, $L_c = 8$, $L_x = 36$

**Metastability between two different phases in which the symmetry breaks in two different ways cold start (above) and hot start (below) where one interface forms.**



A similar pattern of phase transitions is observed if we change the magnetic field at fixed compactification radius, remember:

$$qB = \frac{2\pi b}{(L_x L_c a^2)}$$




We start with a center-translation symmetric system

(three center domains and three interfaces)

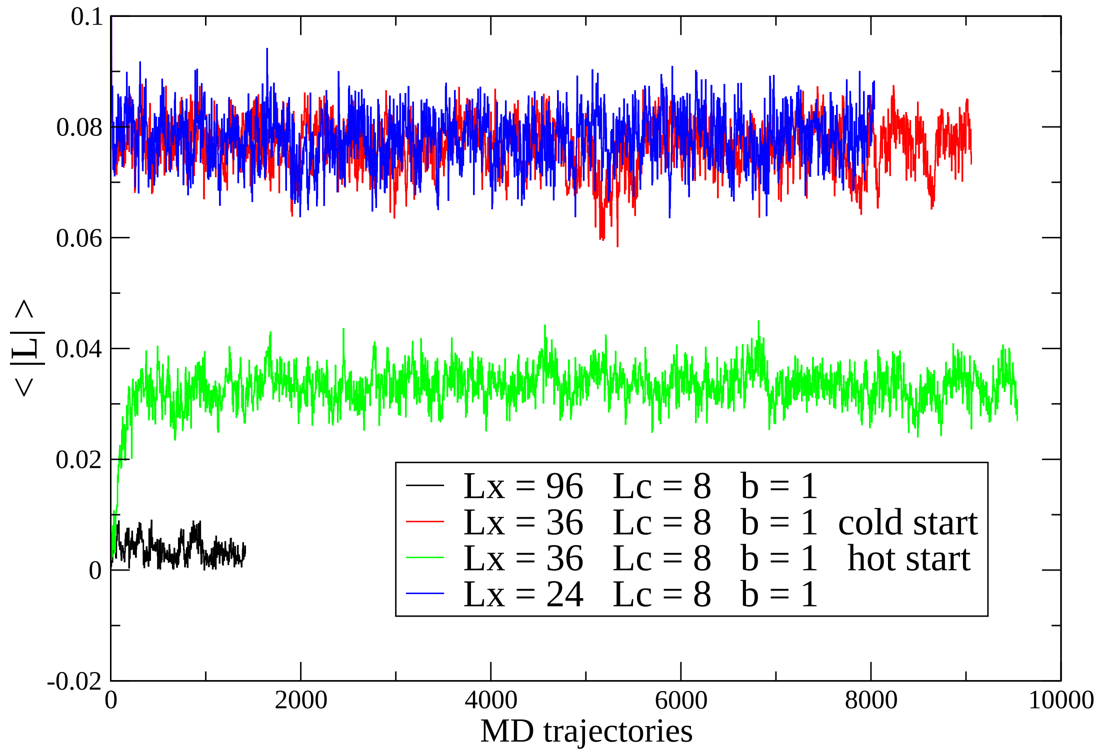
then, as we increase $B$ we go through a phase with just two center domains

(banana phase?)

and finally to a phase with a single center sector.

**Going from one phase to the other takes place through first order phase transitions at which interfaces are destroyed or created.**

**This is clearly visible from time histories of the global Polyakov loop (averaged over whole space) which shows strong metastabilities**



# Conclusions

We have shown that non-trivial phases appears, with different "crystalline" structures, and with discontinous transitions between them.

The complete theory (non-degenerate quarks with different electric charges) will have an even less trivial phase structures

Such phenomenology could be interesting for theories with compactified extra-dimensions in the presence of a background field

It is an interesting deformation of standard QCD by itself. For instance, what is the fate and the dynamics of different topological objects in the different phases, i.e. in the different center domain lattices?