Dualization of non-abelian lattice gauge theories

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Work done together with Carlotta Marchis ...

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Introductory comments

Dualization

- Here the term "dualization" is used for an exact rewriting of the partition sum (and observables) in terms of new variables.
- The dual variables are world-lines for matter fields and world-sheets for gauge fields.
- Originally the dualization program was motivated by mapping systems with complex action problem to a real and positive representation.
- Dual representations have other interesting and beautiful properties. In particular the particle number is mapped to a winding number of worldlines in the dual representation.
- Nice example: Analyze distribution in sector with winding number 2 $\Rightarrow$ scattering data.
- Goal of this project: Explore techniques and applications for dual variables further, in particular for non-abelian gauge theories.
- A key problem is the re-ordering of the non-abelian gauge variables.
- Here we use a new approach: We decompose the gauge action into abelian color cycles (= paths through color space along plaquettes) that solve the re-ordering problem.

How does dualization of U(1) LGT work?

- Partition sum:

$$Z \;=\; \int D[U]\; e^{\,\beta \sum_{x,\mu<\nu} \text{Re}\, U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^\star_{x+\hat{\nu},\mu}\, U^\star_{x,\nu}} \quad , \quad \int D[U] \;=\; \prod_{x,\mu} \int_{U(1)} dU_{x,\mu}$$

- Expansion of the Boltzmann factor:

$$Z = \int D[U] \prod_{x,\mu<\nu} e^{\,\frac{\beta}{2}\, U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^\star_{x+\hat{\nu},\mu}\, U^\star_{x,\nu}}\; e^{\,\frac{\beta}{2}\, U^\star_{x,\mu}\, U^\star_{x+\hat{\mu},\nu}\, U_{x+\hat{\nu},\mu}\, U_{x,\nu}} \;=$$

$$\int D[U] \prod_{x,\mu<\nu} \sum_{p_{x,\mu\nu}} \sum_{\overline{p}_{x,\mu\nu}} \frac{\left(\frac{\beta}{2}\right)^{p_{x,\mu\nu}+\overline{p}_{x,\mu\nu}}}{p_{x,\mu\nu}!\;\overline{p}_{x,\mu\nu}!} \Big(U_{x,\mu} U_{x+\hat{\mu},\nu} U^\star_{x+\hat{\nu},\mu} U^\star_{x,\nu}\Big)^{p_{x,\mu\nu}} \Big(U^\star_{x,\mu} U^\star_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu} U_{x,\nu}\Big)^{\overline{p}_{x,\mu\nu}}$$

- Reordering the terms (use: $U^\star_{x,\mu} = U^{-1}_{x,\mu}$):

$$Z \;=\; \sum_{\{p,\overline{p}\}} \prod_{x,\mu<\nu} \frac{\left(\frac{\beta}{2}\right)^{p_{x,\mu\nu}+\overline{p}_{x,\mu\nu}}}{p_{x,\mu\nu}!\;\overline{p}_{x,\mu\nu}!} \prod_{x,\mu} \int_{U(1)} dU_{x,\mu}\, \Big(U_{x,\mu}\Big)^{\sum_{\nu:\mu<\nu}[d_{x,\mu\nu}-d_{x-\hat{\nu},\mu\nu}] - \sum_{\rho:\rho<\mu}[d_{x,\rho\mu}-d_{x-\hat{\rho},\rho\mu}]}$$

$$d_{x,\mu\nu} \equiv p_{x,\mu\nu} - \overline{p}_{x,\mu\nu}$$

How does dualization of U(1) LGT work?

- Integrating out the gauge fields ($\int dU\, U^n = \delta(n)$):

$$Z = \sum_{\{p,\overline{p}\}} \prod_{x,\mu<\nu} \frac{\left(\frac{\beta}{2}\right)^{p_{x,\mu\nu}+\overline{p}_{x,\mu\nu}}}{p_{x,\mu\nu}!\,\overline{p}_{x,\mu\nu}!} \prod_{x,\mu} \delta(\, J_{x,\mu}\,)$$

$$J_{x,\mu} = \sum_{\nu:\mu<\nu} [d_{x,\mu\nu} - d_{x-\hat{\nu},\mu\nu}] - \sum_{\rho:\rho<\mu} [d_{x,\rho\mu} - d_{x-\hat{\rho},\rho\mu}]\,, \quad d_{x,\mu\nu} \equiv p_{x,\mu\nu} - \overline{p}_{x,\mu\nu}$$

  The partition function is exactly rewritten into a sum over configurations of the plaquette occupation numbers $p_{x,\mu\nu}, \overline{p}_{x,\mu\nu} \in \mathbb{N}_0$, which obey constraints giving rise to an interpretation as a sum over worldsheets.

- Key to success was the reordering of the abelian gauge links:

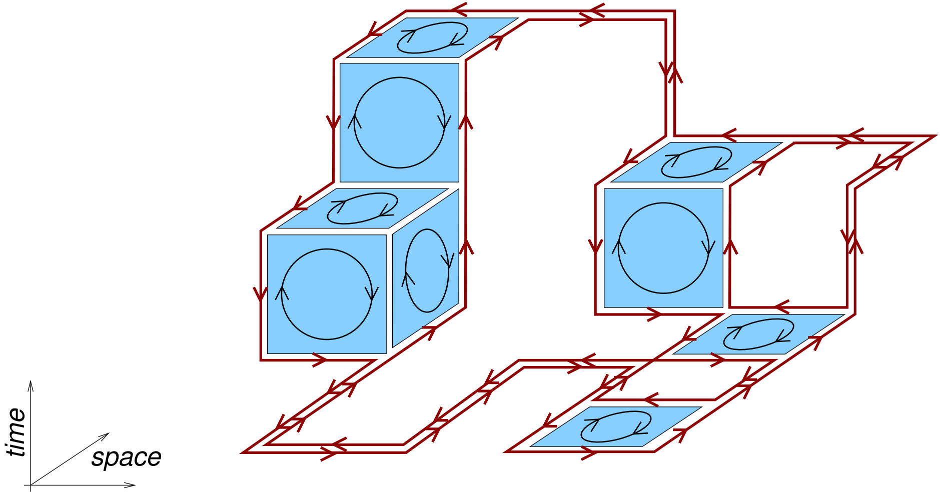
$$\prod_{x,\mu<\nu} \left(U_{x,\mu}U_{x+\hat{\mu},\nu}U^\star_{x+\hat{\nu},\mu}U^\star_{x,\nu}\right)^{p_{x,\mu\nu}} \left(U^\star_{x,\mu}U^\star_{x+\hat{\mu},\nu}U_{x+\hat{\nu},\mu}U_{x,\nu}\right)^{\overline{p}_{x,\mu\nu}}$$

$$= \prod_{x,\mu} \left(U_{x,\mu}\right)^{\sum_{\nu:\mu<\nu}[d_{x,\mu\nu}-d_{x-\hat{\nu},\mu\nu}]-\sum_{\rho:\rho<\mu}[d_{x,\rho\mu}-d_{x-\hat{\rho},\rho\mu}]}$$

  Not possible for non-abelian theories!

Dual variables = worldsheets coupled to matter loops

Matter fields appear as loops that serve as boundaries for the gauge worldsheets.
Chemical potential couples to the temporal winding number of the loops.



Delgado Mercado, Gattringer, Schmidt. Phys.Rev.Lett. 111 (2013) 141601

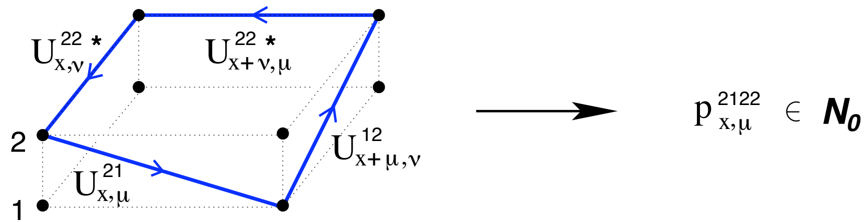
Abelian Color Cycles (ACCs)

Decomposition of the non-abelian action into abelian color cycles:

- Action for SU(2) lattice gauge theory ($U_{x,\mu} \in$ SU(2)) :

$$S \;=\; -\frac{\beta}{2}\sum_{x,\mu<\nu} \text{Tr}\; U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^{\dagger}_{x+\hat{\nu},\mu}\, U^{\dagger}_{x,\nu} \;=\; -\frac{\beta}{2}\sum_{x,\mu<\nu}\;\sum_{a,b,c,d=1}^{2} U^{ab}_{x,\mu}\, U^{bc}_{x+\hat{\mu},\nu}\, U^{dc\,\star}_{x+\hat{\nu},\mu}\, U^{ad\,\star}_{x,\nu}$$

- The products of link components $U^{ab}_{x,\mu}\, U^{bc}_{x+\hat{\mu},\nu}\, U^{dc\,\star}_{x+\hat{\nu},\mu}\, U^{ad\,\star}_{x,\nu}$ are the Abelian Color Cycles (ACCs) (= paths through color space along plaquettes) we use for expanding the Boltzmann factor. Example:



- Suitable parameterization:

$$U_{x,\mu} \;=\; \begin{bmatrix} \cos\theta_{x,\mu}\, e^{i\alpha_{x,\mu}} & \sin\theta_{x,\mu}\, e^{i\beta_{x,\mu}} \\ -\sin\theta_{x,\mu}\, e^{-i\beta_{x,\mu}} & \cos\theta_{x,\mu}\, e^{-i\alpha_{x,\mu}} \end{bmatrix} \qquad \theta_{x,\mu} \in [0,\pi/2]\;,\quad \alpha_{x,\mu}, \beta_{x,\mu} \in [-\pi,\pi]$$

Expansion in Abelian Color Cycles

- Partition sum:

$$Z = \int D[U] \; \exp\left(\frac{\beta}{2} \sum_{x,\mu<\nu} \sum_{a,b,c,d} U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc\,\star}_{x+\hat{\nu},\mu} \, U^{ad\,\star}_{x,\nu} \right) \; , \quad \int D[U] \; = \prod_{x,\mu} \int_{SU(2)} dU_{x,\mu}$$

- Expansion of the Boltzmann factor:

$$\begin{aligned} Z \; &= \; \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} e^{\,\frac{\beta}{2} \, U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc\,\star}_{x+\hat{\nu},\mu} \, U^{ad\,\star}_{x,\nu}} \\ &= \; \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} \; \sum_{p^{abcd}_{x,\mu\nu}=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}\,!} \left(U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc\,\star}_{x+\hat{\nu},\mu} \, U^{ad\,\star}_{x,\nu} \right)^{p^{abcd}_{x,\mu\nu}} \end{aligned}$$

- Reordering the terms:

$$Z \; = \sum_{\{p\}} \prod_{x,\mu<\nu} \prod_{a,b,c,d} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}\,!} \prod_{x,\mu} \int d_H[\theta_{x,\mu}, \alpha_{x,\mu}, \beta_{x,\mu}] \prod_{ab} \left(U^{ab}_{x,\mu} \right)^{N^{ab}_{x,\mu}[p]} \left(U^{ab\,\star}_{x,\mu} \right)^{\overline{N}^{ab}_{x,\mu}[p]}$$

Remaining link integrals can be solved and give constraints and weights for the configurations $\{p\}$ of the cycle occupation numbers $p^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$.

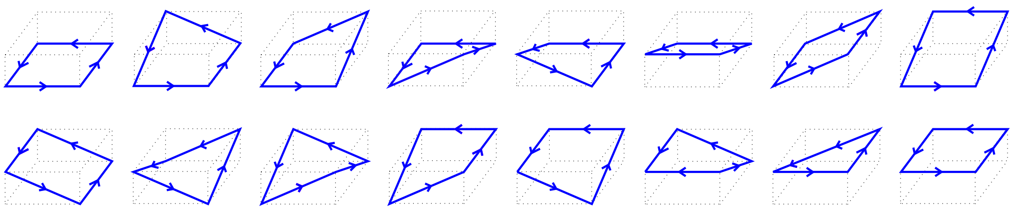
Partition function as sum over occupation numbers of Abelian Color Cycles

- Dual partition sum:

$$Z = \sum_{\{p\}} W_\beta[p] \, (-1)^{\sum_{x,\mu} J^{21}_{x,\mu}} \prod_{x,\mu<\nu} \delta\left(J^{11}_{x,\mu} - J^{22}_{x,\mu}\right) \, \delta\left(J^{12}_{x,\mu} - J^{21}_{x,\mu}\right)$$

$$J^{ab}_{x,\mu} = \text{ total flux from } a \text{ to } b \text{ along the link } x,\mu$$

- 16 possible cycles that can be occupied (i.e., $p^{abcd}_{x,\mu\nu} > 0$):



- Constraints at each link:

Adding matter

Staggered fermions in an SU(2) background

- Fermionic partition sum:

$$Z_F[U] = \int D[\overline{\psi}, \psi] \; e^{-S_F[\overline{\psi},\psi,U]}$$

$$\overline{\psi}_x = (\, \overline{\psi}_x^{\,1}, \overline{\psi}_x^{\,2} \,) \; , \quad \psi_x = \begin{pmatrix} \psi_x^1 \\ \\ \psi_x^2 \end{pmatrix}$$

- Action and its decomposition into color bilinears:

$$S_F[\overline{\psi}, \psi, U] = \sum_x \Big[m \, \overline{\psi}_x \psi_x + \sum_\mu \frac{\gamma_{x,\mu}}{2} \Big(\overline{\psi}_x U_{x,\mu} \psi_{x+\hat{\mu}} - \overline{\psi}_{x+\hat{\mu}} U^\dagger_{x,\mu} \psi_x \Big) \Big]$$

$$= \sum_x \Big[m \sum_a \overline{\psi}_x^{\,a} \psi_x^a + \sum_\mu \frac{\gamma_{x,\mu}}{2} \sum_{a,b} \Big(\overline{\psi}_x^{\,a} U_{x,\mu}^{ab} \psi_{x+\hat{\mu}}^b - \overline{\psi}_{x+\hat{\mu}}^{\,b} U_{x,\mu}^{ab\,\star} \psi_x^a \Big) \Big]$$

Loop expansion

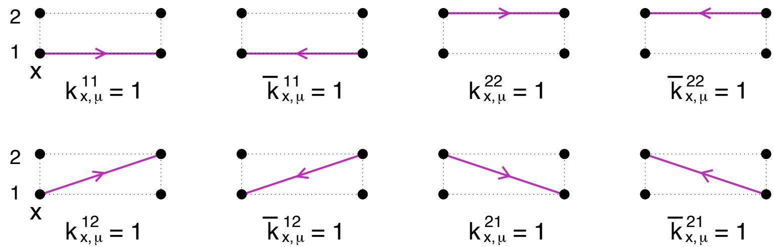
- Expanding the Boltzmann factors in the fermionic partition sum:

$$
\begin{aligned}
Z_F[U] &= \int D[\overline{\psi},\psi] \prod_x \prod_a e^{-m\overline{\psi}^a_x \psi^a_x} \prod_{x,\mu} \prod_{a,b} e^{-\frac{\gamma_{x,\mu}}{2}\overline{\psi}^a_x U^{ab}_{x,\mu}\psi^b_{x+\hat{\mu}}}\, e^{\frac{\gamma_{x,\mu}}{2}\overline{\psi}^b_{x+\hat{\mu}} U^{ab\star}_{x,\mu}\psi^a_x} \\
&= \int D[\overline{\psi},\psi] \prod_x \prod_a \sum_{s^a_x=0}^{1} (-m\overline{\psi}^a_x \psi^a_x)^{s^a_x} \\
&\quad \times \prod_{x,\mu} \prod_{a,b} \sum_{k^{ab}_{x,\mu}=0}^{1} \Big(-\frac{\gamma_{x,\mu}}{2}\overline{\psi}^a_x U^{ab}_{x,\mu}\psi^b_{x+\hat{\mu}}\Big)^{k^{ab}_{x,\mu}} \sum_{\overline{k}^{ab}_{x,\mu}=0}^{1} \Big(\frac{\gamma_{x,\mu}}{2}\overline{\psi}^b_{x+\hat{\mu}} U^{ab\star}_{x,\mu}\psi^a_x\Big)^{\overline{k}^{ab}_{x,\mu}} \\
&= \frac{1}{2^{2V}} \sum_{\{s,k,\overline{k}\}} (2m)^{\sum_{x,a} s^a_x} \prod_{x,\mu} \prod_{a,b} (U^{ab}_{x,\mu})^{k^{ab}_{x,\mu}} (U^{ab\star}_{x,\mu})^{\overline{k}^{ab}_{x,\mu}} \\
&\quad \times \int D[\overline{\psi},\psi] \prod_x \prod_a (\overline{\psi}^a_x \psi^a_x)^{s^a_x} \prod_{x,\mu} \prod_{a,b} (-\gamma_{x,\mu}\overline{\psi}^a_x \psi^b_{x+\hat{\mu}})^{k^{ab}_{x,\mu}} (\gamma_{x,\mu}\overline{\psi}^b_{x+\hat{\mu}} \psi^a_x)^{\overline{k}^{ab}_{x,\mu}}
\end{aligned}
$$

The Grassmann integral is saturated by monomers ($s^a_x = 1$), dimers ($k^{ab}_{x,\mu} = \overline{k}^{ab}_{x,\mu} = 1$) and loops of $k^{ab}_{x,\mu} = 1$ and $\overline{k}^{ab}_{x,\mu} = 1$. Only loops introduce signs!

Interaction with the gauge fields

- $k^{ab}_{x,\mu}$ and $\overline{k}^{ab}_{x,\mu}$ introduce color flux on the links:



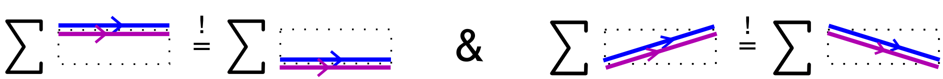
- Full partition sum:

$$Z \;=\; \sum_{\{p,k,\overline{k},s\}} C_{MDL}[s,k,\overline{k}\,]\; W_\beta[p]\; W_m[s] \;\prod_{x,\mu}(-1)^{\,J^{21}_{x,\mu}+k^{21}_{x,\mu}+\overline{k}^{21}_{x,\mu}} \;\prod_{L} \text{sign}\,(L)$$

$$\times \prod_{x,\mu<\nu} \delta\Big(J^{11}_{x,\mu}+k^{11}_{x,\mu}-\overline{k}^{11}_{x,\mu} - [J^{22}_{x,\mu}+k^{22}_{x,\mu}-\overline{k}^{22}_{x,\mu}]\Big)\; \delta\Big(J^{12}_{x,\mu}+k^{12}_{x,\mu}-\overline{k}^{12}_{x,\mu} - [J^{21}_{x,\mu}+k^{21}_{x,\mu}-\overline{k}^{21}_{x,\mu}]\Big)$$

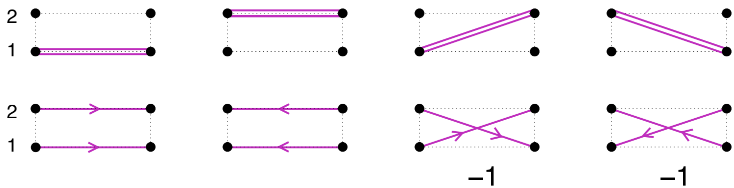
$$\text{sign}\,(L) \;=\; -\,(-1)^{\#\,plaquettes}\,(-1)^{length/2}\,(-1)^{temp.winding}$$

- Gauge constraints:

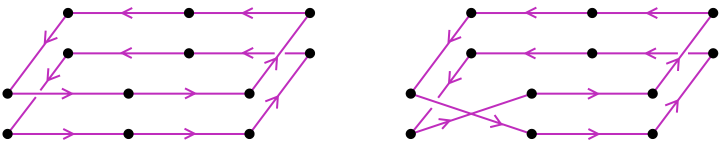


Strong coupling loops ($\beta = 0$)

- Strong coupling: $\beta = 0 \Rightarrow$ only fermion lines. The gauge constraints limit the number of admissible link elements:



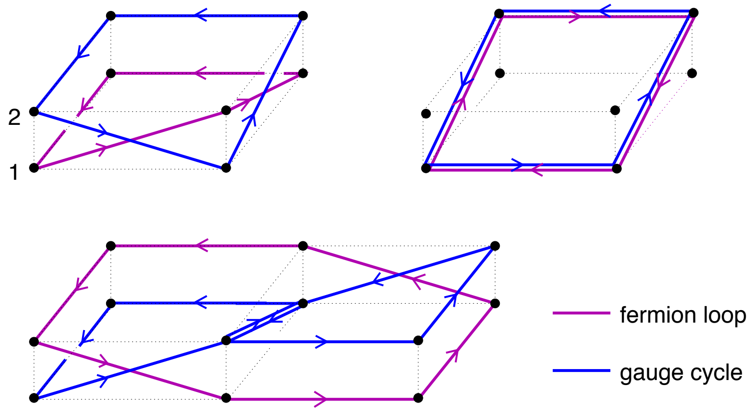
- Signs from color flips compensate fermion loop signs:



- In the strong coupling limit all contributions are positive.

Loops at $\beta > 0$

- At $\beta > 0$ one can activate ACCs to satisfy the gauge constraints for more general fermion loops. Examples of $\mathcal{O}(\beta)$ and $\mathcal{O}(\beta^2)$:



- One can show that all contributions up to $\mathcal{O}(\beta^3)$ are positive. From $\mathcal{O}(\beta^4)$ some configurations with negative signs appear. Chemical potential couples to total temporal winding number.

First look at the ACC construction for SU(3) lattice gauge theory

ACCs for SU(3):

- Action for SU(3) lattice gauge theory ($U_{x,\mu} \in$ SU(3)) :

$$S_G[U] = -\frac{\beta}{3} \sum_{x,\mu<\nu} \text{Re Tr}\, U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^{\dagger}_{x+\hat{\nu},\mu}\, U^{\dagger}_{x,\nu}$$

$$= -\frac{\beta}{6} \sum_{x,\mu<\nu} \sum_{a,b,c,d=1}^{3} \Big[U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu} + U^{ab\,\star}_{x,\mu} U^{bc\,\star}_{x+\hat{\mu},\nu} U^{dc}_{x+\hat{\nu},\mu} U^{ad}_{x,\nu} \Big]$$

- Complex conjugate pairs $\Rightarrow$ ACCs come in both orientations

- Similar to U(1) we work with two sets of expansion indices:

$$p^{abcd}_{x,\mu\nu} \in \mathbb{Z} \qquad l^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$$

Parameterization of SU(3):

- Parameterization of SU(3) gauge links (J.B. Brozan, PRD 1988):

$$\begin{bmatrix} c_1 c_2\, e^{i\phi_1} & s_1\, e^{i\phi_3} & c_1 s_2\, e^{i\phi_4} \\ s_2 s_3\, e^{-i\phi_4 - i\phi_5} - s_1 c_2 c_3\, e^{i\phi_1 + i\phi_2 - i\phi_3} & c_1 c_3\, e^{i\phi_2} & -c_2 s_3\, e^{-i\phi_1 - i\phi_5} - s_1 s_2 c_3\, e^{i\phi_2 - i\phi_3 + i\phi_4} \\ -s_2 c_3\, e^{-i\phi_2 - i\phi_4} - s_1 c_2 s_3\, e^{i\phi_1 - i\phi_2 + i\phi_5} & c_1 s_3\, e^{i\phi_5} & c_2 c_3\, e^{-i\phi_1 - i\phi_2} - s_1 s_2 s_3\, e^{-i\phi_3 + i\phi_4 + i\phi_5} \end{bmatrix}$$

$$c_i = \cos\theta_i,\ s_i = \sin\theta_i,\ \theta_i \in [0, \pi/2] \quad , \quad \phi_i \in [0, 2\pi]$$

- Some matrix elements are sums of two terms ...

$\Rightarrow$ additional auxiliary variables that are later absorbed in weights

Dual partition sum:

- Dual partition sum:

$$Z \;=\; \sum_{\{p,l\}} W_\beta[p,l]\;(-1)^{\sum_{x,\mu} J^{12}_{x,\mu}+J^{23}_{x,\mu}+J^{31}_{x,\mu}}$$
$$\times \prod_{x,\mu} \delta(J^{11}_{x,\mu}+J^{12}_{x,\mu}-J^{33}_{x,\mu}-J^{23}_{x,\mu})\;\delta(J^{22}_{x,\mu}+J^{12}_{x,\mu}-J^{33}_{x,\mu}-J^{31}_{x,\mu})$$
$$\delta(J^{13}_{x,\mu}+J^{12}_{x,\mu}-J^{31}_{x,\mu}-J^{21}_{x,\mu})\;\delta(J^{32}_{x,\mu}+J^{12}_{x,\mu}-J^{23}_{x,\mu}-J^{21}_{x,\mu})$$

- Cycle occupation numbers: $p^{abcd}_{x,\mu\nu} \in \mathbb{Z} \qquad l^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$

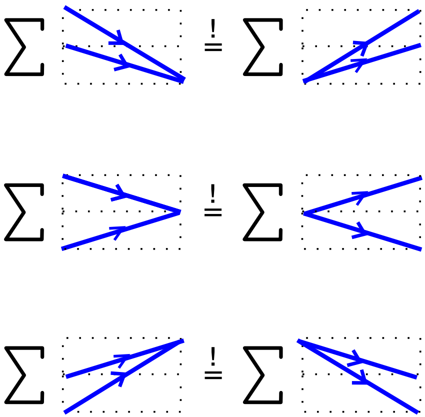
- Constraints involve only the $p^{abcd}_{x,\mu\nu}$ via:

$$J^{ab}_{x,\mu} \;=\; \text{total } p\text{-flux from } a \text{ to } b \text{ along the link } x,\mu$$

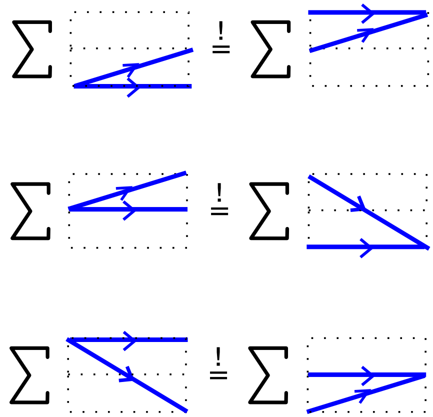
Structure of constraints for SU(3):

The constraints can be combined into the following flux rules:

Conservation of flux:



Exchange of flux:



Summary

- The action of non-abelian gauge fields is decomposed into Abelian Color Cycles (= paths through color space along plaquettes).
- Expanding the Boltzmann factor introduces an occupation number for each cycle.
- The link contributions to the cycles are $\mathbb{C}$-valued and the re-ordering problem is solved.
- The original link-degrees of freedom can be integrated out in closed form. This generates weights, constraints and signs for configurations of cycle occupation numbers.
- The cycle construction can be generalized by including matter fields.
- Weights for all terms of the strong coupling expansion are known in closed form.
- Up to $\mathcal{O}(\beta^3)$ only positive terms contribute.
- Generalization of the abelian cycle decomposition to other gauge groups is possible
  $\Rightarrow$ first steps for SU(3) LGT.