Harmonic oscillator force in heavy quarkonia

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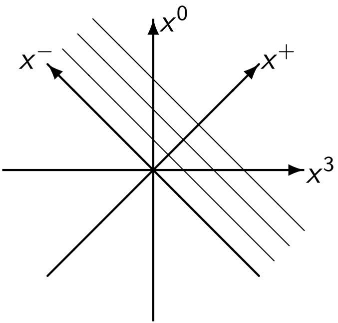
Outline

Outline

Front form of Hamiltonian dynamics



$$x^+ = x^0 + x^3$$
$$x^- = x^0 - x^3$$

$$p_\mu x^\mu = \frac{1}{2} p^- x^+ + \frac{1}{2} p^+ x^- - p^1 x^1 - p^2 x^2 \,.$$

$$p^- = \frac{m^2 + (p^\perp)^2}{p^+} \,,$$

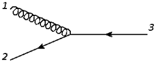
$$\mathcal{P}^- = \frac{m^2 + (p^\perp)^2}{p^+} + \frac{\mu^2 + (k^\perp)^2}{k^+} = \frac{\mathcal{M}^2(x,\kappa) + (\mathcal{P}^\perp)^2}{\mathcal{P}^+} \,,$$

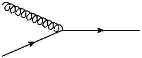
$$\mathcal{M}^2(x,\kappa) = \frac{m^2 + \kappa^2}{1-x} + \frac{\mu^2 + \kappa^2}{x} \,,$$

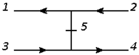
$$x = \frac{k^+}{p^+ + k^+} \,, \quad \kappa^\perp = \frac{p^+ k^\perp - k^+ p^\perp}{p^+ + k^+} \,.$$

Canonical FF Hamiltonian of QCD

$$
\begin{aligned}
H_{\text{can}} &= \frac{m^2 + (k_1^\perp)^2}{k_1^+} b_1^\dagger b_1 + \frac{m^2 + (k_1^\perp)^2}{k_1^+} d_1^\dagger d_1 + \frac{(k_1^\perp)^2}{k_1^+} a_1^\dagger a_1 \\
&+ g \left(\tilde{\delta}_{21.3}\ t^1_{23}\ \bar{u}_2\ \not{\varepsilon}^*_1 u_3\ b_2^\dagger a_1^\dagger b_3 - \tilde{\delta}_{21.3}\ t^1_{32}\ \bar{v}_3\ \not{\varepsilon}^*_1 v_2\ d_2^\dagger a_1^\dagger d_3 + \ h.c. \right) \\
&- g^2 \tilde{\delta}_{13.42} \sum_{i_5} t^5_{12} t^5_{43}\ \frac{1}{(k_5^+)^2} j^+_{12} \bar{j}^+_{43}\ b_1^\dagger d_3^\dagger d_4 b_2 \ + \dots \\
&= E_1 \left(b_1^\dagger b_1 + d_1^\dagger d_1 + a_1^\dagger a_1 \right)
\end{aligned}
$$

$+ \ g \Big($ 

$+$  $+ \ h.c. \Big)$

$+ \ g^2$ 

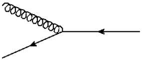
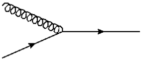
$+ \dots$

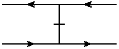
Canonical Hamiltonian is not well defined. We need to regularize it.

Regularization I

We do not regularize fields but rather interactions.

$$H = E_1 \left(b_1^\dagger b_1 + d_1^\dagger d_1 + a_1^\dagger a_1 \right)$$

$$+ \; g \left(r_{21.3} \right.$$  $$+ \; r_{21.3}$$  $$\left. + \; h.c. \right)$$

$$+ \; g^2 r_{C\,13.42}$$  $$+ \dots$$

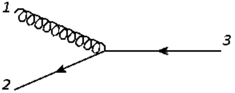
For every quark operator in the interaction vertex we write a factor

$$e^{-\frac{m^2+\kappa^2}{x\Delta^2}}$$

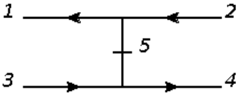
and for every gluon operator in the vertex we write

$$e^{-\frac{\delta^2+\kappa^2}{x\Delta^2}} \; .$$

Regularization II



$$r_{21.3} = e^{-\frac{m^2+\kappa_{2/3}^2}{x_{2/3}\Delta^2}} e^{-\frac{\delta^2+\kappa_{1/3}^2}{x_{1/3}\Delta^2}} e^{-\frac{m^2}{\Delta^2}} = e^{-\frac{\mathcal{M}_{21\,\delta}^2+m^2}{\Delta^2}} ,$$



$$r_{C13.42} \quad = \quad \theta(z)\ r_{25.1} r_{35.4} + \theta(-z)\ r_{45.3} r_{15.2} ,$$

Renormalization Group Procedure for Effective Particles I

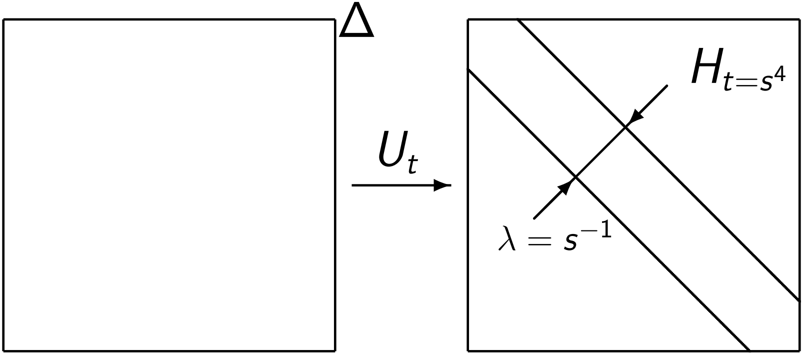


Figure: Initial theory → effective theory. RGPEP introduces form factors into vertices, which narrow the Hamiltonian in free-states basis.

Renormalization Group Procedure for Effective Particles II

- The narrowing of the Hamiltonian is realized by the following differential equation,

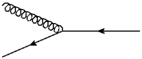
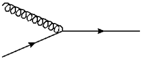
$$\frac{d}{dt}\mathcal{H}_t = [[\mathcal{H}_f, \mathcal{H}_{Pt}], \mathcal{H}_t] \ .$$

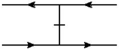
- The generation of transformation U_t is therefore, different compared to the one used in Phys.Rev. D69 (2004) 065002, it allows for nonperturbative calculations.
- But before nonperturbative calculations will be performed we write perturbative expansion of the solution first,

$$H_t \quad = \quad H_f + g\, H_{t\,1} + g^2\, H_{t\,2} + \ldots$$

Renormalized Hamiltonian

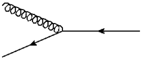
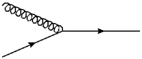
$$H = E_1 \left(b_1^\dagger b_1 + d_1^\dagger d_1 + a_1^\dagger a_1 \right)$$

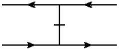
$+ \; g \Big(r_{21.3}$  $+ \; r_{21.3}$  $+ \; h.c. \Big)$

$+ \; g^2 r_{C\,13.42}$  $+ \ldots$

Renormalized Hamiltonian

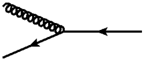
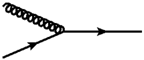
$$H_t = E_1 \left(b^\dagger_{1\,t} b_{1\,t} + d^\dagger_{1\,t} d_{1\,t} + a^\dagger_{1\,t} a_{1\,t} \right)$$

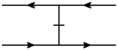
$+ \; g \Big(r_{21.3}$  $+ \; r_{21.3}$  $+ \; h.c. \Big)$

$+ \; g^2 r_{C\,13.42}$ 

Renormalized Hamiltonian

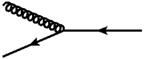
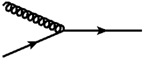
$$H_t = E_1 \left(b^\dagger_{1\,t} b_{1\,t} + d^\dagger_{1\,t} d_{1\,t} + a^\dagger_{1\,t} a_{1\,t} \right)$$

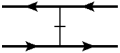
$$+ \; g \left(r_{21.3} f_{t\,21.3}\right.$$  $$+ \; r_{21.3} f_{t\,21.3}$$  $$\left. + \; h.c. \right)$$

$$+ \; g^2 r_{C\,13.42}$$ 

Renormalized Hamiltonian

$$H_t = E_1 \left(b^\dagger_{1\,t} b_{1\,t} + d^\dagger_{1\,t} d_{1\,t} + a^\dagger_{1\,t} a_{1\,t} \right)$$

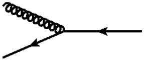
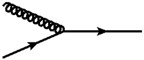
$$+ \; g \Big(r_{21.3} f_{t\,21.3}$$  $$+ \; r_{21.3} f_{t\,21.3}$$  $$+ \; h.c. \Big)$$

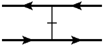
$$+ \; g^2 r_{C\,13.42} f_{t\,13.24}$$ 

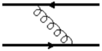
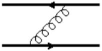
$$f_{t\,21.3} = e^{-t(\mathcal{M}^2_{21} - m^2)^2} \,, \quad f_{t\,13.24} = e^{-t(\mathcal{M}^2_{13} - \mathcal{M}^2_{24})^2} \,,$$

Renormalized Hamiltonian

$$H_t = E_1 \left(b^\dagger_{1\,t} b_{1\,t} + d^\dagger_{1\,t} d_{1\,t} + a^\dagger_{1\,t} a_{1\,t} \right)$$

$$+ \; g \Big(r_{21.3} f_{t\,21.3}$$  $$+ \, r_{21.3} f_{t\,21.3}$$  $$+ \; h.c. \Big)$$

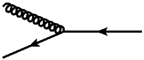
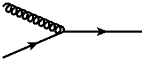
$$+ \; g^2 r_{C\,13.42} f_{t\,13.24}$$ 

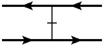
$$+ \; g^2 r_{25.1} r_{35.4} \mathcal{F}_\Sigma$$  $$+ \, g^2 r_{45.3} r_{15.2} \mathcal{F}_Z$$ 

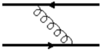
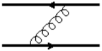
$$f_{t\,21.3} = e^{-t(\mathcal{M}^2_{21} - m^2)^2} \,, \quad f_{t\,13.24} = e^{-t(\mathcal{M}^2_{13} - \mathcal{M}^2_{24})^2} \,,$$

Renormalized Hamiltonian

$$H_t = E_1 \left(b^\dagger_{1\,t} b_{1\,t} + d^\dagger_{1\,t} d_{1\,t} + a^\dagger_{1\,t} a_{1\,t} \right)$$

$+ \; g \Big(r_{21.3} f_{t\,21.3}$  $+ \; r_{21.3} f_{t\,21.3}$  $+ \; h.c. \Big)$

$+ \; g^2 r_{C\,13.42} f_{t\,13.24}$ 

$+ \; g^2 r_{25.1} r_{35.4} \mathcal{F}_{\Sigma}$  $+ \; g^2 r_{45.3} r_{15.2} \mathcal{F}_{Z}$ 

$+ \; g^2$  $+$ 

$$f_{t\,21.3} = e^{-t(\mathcal{M}^2_{21} - m^2)^2} , \quad f_{t\,13.24} = e^{-t(\mathcal{M}^2_{13} - \mathcal{M}^2_{24})^2} ,$$

Renormalized Hamiltonian* I

Mass terms

The quark mass term in H_{t2} consists of effective part,

$$(H_{mQ\,t})_{1.2} = \frac{4}{3}\delta_{c_1c_2}\tilde{\delta}_{1.2}\int [56] k_1^+\ \tilde{\delta}_{56.1}\ r_{65.1}^2 \frac{f_{t\,65.1}^2 - 1}{\mathcal{M}_{56}^2 - m^2}\sum_{\sigma_6} d_{\mu\nu} j_{16}^{\mu} j_{62}^{\nu}\ ,$$

and the UV counterterm (which cancels divergence of $H_{mQ\,t}$ coming from $\kappa \to \infty$),

$$(H_{mQ}^*)_{1.2} = \frac{4}{3}\delta_{c_1c_2}\tilde{\delta}_{1.2}\int [56] k_2^+\ \tilde{\delta}_{56.2}\ r_{65.1} r_{65.2} \frac{1}{\mathcal{M}_{56}^2 - m^2}\sum_{\sigma_6} d_{\mu\nu} j_{16}^{\mu} j_{62}^{\nu}\ ,$$

However, small-x divergences remain,

$$H_{mQ\,t} \sim \sqrt{\frac{2\pi}{t}}\log\frac{\Delta}{\delta}\ .$$

Analogous expressions can be written for antiquark mass term.

Renormalized Hamiltonian* II

Four-body vertex interaction term

The exchange part of $H_{t\,2}$ consists of the term derived from instantaneous interactions in the initial Hamiltonian,

$$(H_{C\,Q\bar{Q}\,t})_{13.42} = C_{13.42}\ r_{C13.42} = -\,\tilde{\delta}_{13.42}\sum_{i_5} t^5_{12}t^5_{43}\ \frac{1}{(k_5^+)^2} j^+_{12}\bar{j}^+_{43}\ r_{C13.42}\ ,$$

and term derived from products of first order initial Hamiltonian vertices,

$$\begin{aligned}(H_{X\,Q\bar{Q}\,t})_{13.42} &= -\,\tilde{\delta}_{13.42}\sum_{i_5} t^5_{12}t^5_{43}\frac{d_{\mu\nu}(k_5)}{k_5^+} j^\mu_{12}\bar{j}^\nu_{43} \\ &\times\ [\theta(z)\mathcal{F}_\Sigma\ r_{25.1}r_{35.4} + \theta(-z)\mathcal{F}_Z\ r_{45.3}r_{15.2}]\ ,\end{aligned}$$

$$\begin{aligned}\mathcal{F}_\Sigma &= \frac{k_1^+(m^2-\mathcal{M}^2_{52}) + k_4^+(m^2-\mathcal{M}^2_{53})}{(m^2-\mathcal{M}^2_{52})^2 + (m^2-\mathcal{M}^2_{53})^2 - (\mathcal{M}^2_{13}-\mathcal{M}^2_{42})^2}\left(1-\frac{f_{25.1}f_{35.4}}{f_{13.42}}\right)\ , \\ \mathcal{F}_Z &= \frac{k_3^+(m^2-\mathcal{M}^2_{54}) + k_2^+(m^2-\mathcal{M}^2_{51})}{(m^2-\mathcal{M}^2_{54})^2 + (m^2-\mathcal{M}^2_{51})^2 - (\mathcal{M}^2_{13}-\mathcal{M}^2_{42})^2}\left(1-\frac{f_{15.2}f_{45.3}}{f_{13.42}}\right)\ .\end{aligned}$$

Outline

Bound state problem in the Fock space

Structure of the eigenvalue problem

$$|\Psi_{\mathrm{meson}}\rangle = |Q_t\bar{Q}_t\rangle + |Q_t\bar{Q}_t G_t\rangle + |Q_t\bar{Q}_t G_t G_t\rangle + \ldots \, .$$

$$H_t|\Psi_{\mathrm{meson}}\rangle = \begin{bmatrix} \ldots & \ldots & \ldots \\ \ldots & H_f + g^2 H_{t\,2} & g H_{t\,1} \\ \ldots & g H_{t\,1} & H_f + g^2 H_{t\,2} \end{bmatrix} \begin{bmatrix} \ldots \\ |Q_t\bar{Q}_t G_t\rangle \\ |Q_t\bar{Q}_t\rangle \end{bmatrix} = E_{\mathrm{meson}} \begin{bmatrix} \ldots \\ |Q_t\bar{Q}_t G_t\rangle \\ |Q_t\bar{Q}_t\rangle \end{bmatrix} \, ,$$

Eigenvalue problem with gluon mass ansatz

Introducing the gluon mass terms to model effects of dotted components and interactions, one obtains the corresponding approximate eigenvalue problem,

$$H|\Psi_{\mathrm{meson}}\rangle = \begin{bmatrix} H_f + \mu^2 & gH_1 \\ gH_1 & H_f + g^2 H_2 \end{bmatrix} \begin{bmatrix} |Q\bar{Q}G\rangle \\ |Q\bar{Q}\rangle \end{bmatrix} = E_{\mathrm{meson}} \begin{bmatrix} |Q\bar{Q}G\rangle \\ |Q\bar{Q}\rangle \end{bmatrix} ,$$

where μ^2 denotes the mass-like operators for gluon in the $Q\bar{Q}G$ sector.

$$\mu^2 \sim \int_5 \frac{\mu^2(x_5, \kappa_5)}{k_5^+} a_5^\dagger a_5 \ .$$

Reduction to the lowest Fock sector I

Reduction of the space is performed using operation R,

$$|QQG\rangle \quad = \quad R|QQ\rangle \ .$$

The general formula for effective Hamiltonian in a subspace of states is then

$$H_{\text{eff}} \quad = \quad \frac{1}{\sqrt{P + R^\dagger R}}(P + R^\dagger)H(P + R)\frac{1}{\sqrt{P + R^\dagger R}} \ .$$

Perturbative evaluation of the 2nd-order effective Hamiltonian for the lowest component gives

$$H_{\text{eff}} \quad = \quad H_f + g^2 H_2 + \frac{g^2}{2} H_1 \left(\frac{1}{E_l - H_f - \mu^2} + \frac{1}{E_{l'} - H_f - \mu^2} \right) H_1 \ .$$

Effective $Q\bar{Q}$ eigenvalue equation I

After reduction to $Q\bar{Q}$ sector, the effective eigenvalue equation is

$$H_{t\,\mathrm{eff}}|\psi_{Q\bar{Q}t}\rangle \quad = \quad \frac{M^2 + P^{\perp 2}}{P^+}|\psi_{Q\bar{Q}t}\rangle \ ,$$

or

$$(P^+ H_{t\,\mathrm{eff}} - P^{\perp 2})|\psi_{Q\bar{Q}t}\rangle \quad = \quad M^2|\psi_{Q\bar{Q}t}\rangle \ .$$

The eigenvalue is mass squared instead of mass or the binding energy.

Effective $Q\bar{Q}$ eigenvalue equation II

For the color singlet states we define the $Q\bar{Q}$ wave function in the following manner,

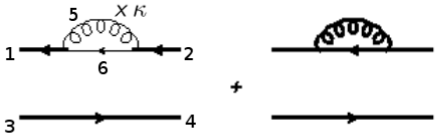
$$|\psi_{Q\bar{Q}t}\rangle = \sum_{c_2c_4}\sum_{\sigma_2\sigma_4}\int[24]\,P^+\,\tilde{\delta}(P-k_2-k_4)\,\frac{\delta_{c_2c_4}}{\sqrt{3}}\,\psi_{\sigma_2\sigma_4}(\kappa_{24}^\perp,x_2)\,b_{2t}^\dagger d_{4t}^\dagger|0\rangle\ .$$

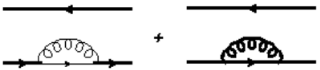
Therefore, the eigenvalue equation becomes,

$$\left(\frac{\kappa_{13}^{\perp 2}+\mathcal{M}_{1,\,t}^2}{x_1}+\frac{\kappa_{13}^{\perp 2}+\mathcal{M}_{3,\,t}^2}{x_3}-M^2\right)\psi_{13}(\kappa_{13}^\perp,x_1)$$
$$+\quad g^2\int[x_2\kappa_{24}^\perp]\ U_{t\,\mathrm{eff}}(13,24)\,\psi_{24}(\kappa_{24}^\perp,x_2) = 0\ ,$$

Effective $Q\bar{Q}$ eigenvalue equation III

$$\mathcal{M}^2_{1,\,t} = m^2 + \frac{4}{3} g^2 \int [x\,\kappa]\, r^2_{65.1}\, f^2_{t\,65.1} \times \sum_{\sigma_6} d_{\mu\nu}(k_5)\, j^{\mu}_{16}\, j^{\nu}_{62} \left(\frac{1}{m^2 - \mathcal{M}^2} - \frac{1}{m^2 - \mathcal{M}^2_{\mu}} \right) ,$$



$\mathcal{M}^2_{3,\,t} =$ 

The subscript μ in $\mathcal{M}^2_{\mu}$ indicates that the gluon has an effective mass.

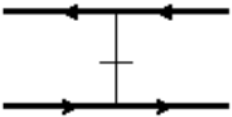
Effective $Q\bar{Q}$ eigenvalue equation IV

The interaction potential $U_{t\,\mathrm{eff}}(13,24)$ contains FF instantaneous interactions, gluon-exchange terms and counterterms:

$$U_{t\,\mathrm{eff}} = H_C + H_{\mathrm{exch}} + H_C^* \ ,$$

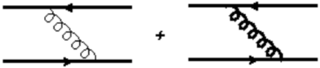
The effective instantaneous vertex acquires only a form factor,

$$H_C = -\frac{4}{3} r_{C\,13.24}\, f_{t\,13.24} \frac{j_{12}^+ \, \bar{j}_{43}^+}{(x_1 - x_2)^2} \ ,$$

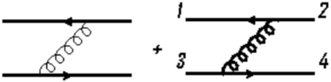


Effective $Q\bar{Q}$ eigenvalue equation V

$$
\begin{aligned}
H_{\mathrm{exch}} &= -\frac{4}{3}\, d_{\mu\nu}(k_5)\, j_{12}^{\mu}\, \bar{j}_{43}^{\nu} \\
&\times \left(\frac{\theta(x_1 - x_2)}{k_5^+}\, r_{25.1} r_{35.4}\, \mathcal{F}_\Sigma + \frac{\theta(x_2 - x_1)}{k_5^+}\, r_{15.2} r_{45.3}\, \mathcal{F}_Z \right) ,
\end{aligned}
$$



$$
\mathcal{F}_\Sigma = f_{t\,13.24}\, \mathcal{F}_\Sigma + f_{t\,4.53}\, f_{t\,1.52}\, \mathcal{R}_\Sigma \ ,
$$

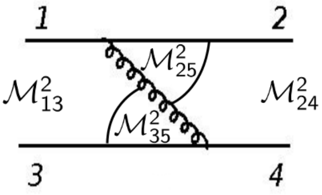


$$
\mathcal{F}_Z = f_{t\,13.24}\, \mathcal{F}_Z + f_{t\,3.54}\, f_{t\,2.51}\, \mathcal{R}_Z \ ,
$$

Effective $Q\bar{Q}$ eigenvalue equation VI

$$\mathcal{F}_\Sigma = \left(1 - \frac{f_{t\,1.52} f_{t\,4.53}}{f_{t\,13.24}}\right) \frac{k_1^+(m^2 - \mathcal{M}_{25}^2) + k_4^+(m^2 - \mathcal{M}_{35}^2)}{(m^2 - \mathcal{M}_{25}^2)^2 + (m^2 - \mathcal{M}_{35}^2)^2 - (\mathcal{M}_{13}^2 - \mathcal{M}_{24}^2)^2} ,$$

$$\mathcal{R}_\Sigma = \frac{1}{2}\left(\frac{k_1^+}{m^2 - \mathcal{M}_{25}^2 - \frac{x_1}{x_5}\mu_{253}^2} + \frac{k_4^+}{m^2 - \mathcal{M}_{53}^2 - \frac{x_4}{x_5}\mu_{253}^2}\right) ,$$



μ_{253}^2 is an ansatz for the gluon-mass function. It depends on relative motion of gluon, 5, with respect to quark, 2, and antiquark, 3, in the intermediate state.

Outline

Small-x divergences I

- Because quantization plane is tangent to a light, cone massless particles can travel in one direction instantaneously. This produces divergences.
- For example, the gluon propagator $d_{\mu\nu}$ contains regular part and singular part in gluon momentum,

$$d_{\mu\nu}(k_5) = -g_{\mu\nu} + \frac{n_\mu k_{5\,\nu} + n_\nu k_{5\,\mu}}{k_5^+} .$$

- Instantaneous interaction term,

$$H_C = -\frac{4}{3} r_{C13.24} f_{t\,13.24} \frac{j_{12}^+ \bar{j}_{43}^+}{x_5^2} .$$

- Mass terms and exchange terms are divergent in the integration region where $x_5 = 0$ and $\kappa_5^\perp = 0$.
- When $\mu = 0$ it was early found out that divergences cancel.

Small-x divergences II

- Terms arising from $\mu \neq 0$ need to be regulated by μ so that every term in the eigenvalue problem is finite when regularization is removed.
- Mass terms are easier to analyze. The divergent integral is

$$\int [x\kappa] r_{65.1}^2 f_{65.1}^2 \frac{\frac{\mu^2}{x}}{\mathcal{M}_\mu^2 - m^2} \frac{4}{x} \ .$$

- The form factor regulates the UV behavior but not the small-x, or large volume, behavior.
- However, if $\mu^2 \sim x^{1+\delta_\mu} \sim \kappa^2 x^{\delta_\mu}$, then the integral is finite.
- For exchange terms we study parts with low- and high-energy exchange separately.

$$\begin{aligned} U_{t\,\mathrm{eff}} &= U_{\mathrm{high}} + U_{\mathrm{low}} + U_g \\ &= \sim (1 - ff) + \sim ff + \sim g_{\mu\nu} \ . \end{aligned}$$

Small-x divergences III

- Low-energy exchange terms are those whose UV behavior is regulated by form factors (in a similar way as in the mass terms).
- Again, the large volume or small-x behavior is not regulated by form factors, but by the gluon mass function.
- The same μ^2 regulates both mass terms and the low-energy exchange terms.
- The high-energy exchange terms do not contain μ^2, but do not lead to divergences in the limit $\delta \to 0$.

One quark eigenvalue problem

- Proper gluon mass ansatz gives well defined, finite, eigenvalue equation for color singlet $Q\bar{Q}$ states.
- One can apply similar reasoning to the eigenproblem of a single quark, where the gluon mass ansatz may be different.

$$
\begin{aligned}
H_t|\Psi_{\rm quark}\rangle &= \left[\begin{array}{cc} H_{f\,t}+\mu_q^2 & gH_{1\,t} \\ gH_{1\,t} & H_{f\,t}+g^2H_{2\,t} \end{array}\right]\left[\begin{array}{c} |Q_tG_t\rangle \\ |Q_t\rangle \end{array}\right] \\
&= E_{\rm quark}\left[\begin{array}{c} |Q_tG_t\rangle \\ |Q_t\rangle \end{array}\right] .
\end{aligned}
$$

- The gluon mass ansatz in this problem can be chosen in such a way that the eigenproblem is divergent.
- This is a possibility how **confinement** could be understood.

Outline

Non-relativistic limit I

- Well defined, finite, eigenproblem for $Q\bar{Q}$ – done.
- Heavy quarks in quarkonia are expected to move slowly so that one can adopt non-relativistic approximation.
- We define non-relativistic relative momenta of quarks,

$$k_{ij}^{\perp} = \frac{1}{2}\frac{\kappa_{ij}^{\perp}}{\sqrt{x_i x_j}} , \quad k_{ij}^{3} = \frac{m}{\sqrt{x_i x_j}}\left(x_i - \frac{1}{2}\right) .$$

where $ij = 13$ or $ij = 24$, and momentum transfer,

$$\vec{q} = \vec{k}_{13} - \vec{k}_{24} .$$

- Formally, the non-relativistic limit means that the quark mass goes to infinity, or $\vec{k}_{ij}/m \to 0$.
- $M^2 = (2m+B)^2 \approx 4m^2 + 4mB$.

Non-relativistic limit II

- Performing the limit we obtain familiar looking Schrodinger equation,

$$\left[\frac{|\vec{k}_{13}|^2}{m} - B + \frac{\delta m^2_{1,\,t}}{2m} + \frac{\delta m^2_{3,\,t}}{2m}\right] \psi_{13}(\vec{k}_{13})$$
$$+ \int \frac{d^3\vec{k}_{24}}{(2\pi)^3} V_{Q\bar{Q}}(\vec{k}_{13} - \vec{k}_{24})\, \psi_{24}(\vec{k}_{24}) \;=\; 0 \;,$$

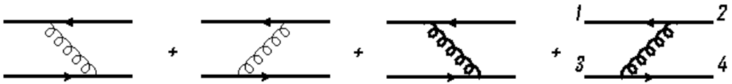
where

$$\frac{\delta m^2_{i,\,t}}{2m} \;=\; \lim_{m\to\infty} \frac{\delta \mathcal{M}^2_{i,\,t}}{2m} \;,$$
$$V_{Q\bar{Q}}(\vec{k}_{13} - \vec{k}_{24}) \;=\; g^2 \lim_{m\to\infty} \left[\frac{1}{4m^2} U_{t\,\text{eff}}(13, 24)\right] \;.$$

Non-relativistic limit III

- If the gluon mass ansatz was zero, the exchange term in the non-relativistic limit would give only Coulomb potential coming from the non-singular part of the gluon propagator (U_g),

$$V_{Q\bar{Q}}(\vec{q}) = -\frac{4}{3}\frac{4\pi\alpha}{|\vec{q}|^2} f_{13.24} \quad (\text{for } \mu = 0) \ .$$



- The Coulomb term is modified by the form factor $f_{13.24}$. The interaction becomes non-local and the effect is visible for distances $\lesssim s \longrightarrow$ the effective particles have nonzero size!

Outline

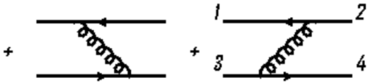
Harmonic oscillator potential I

- For nonzero gluon mass ansatz we have

$$\int \frac{d^3q}{(2\pi)^3} V_{Q\bar{Q}}(\vec{q})\psi(\vec{k}-\vec{q}) \quad = \quad \int \frac{d^3q}{(2\pi)^3} \left[V_C(\vec{q}) + W(\vec{q})\right] \psi(\vec{k}-\vec{q}) \ .$$

where

$$\begin{aligned} W(\vec{q}\,) \quad &= \quad \frac{4}{3}\, 4\pi\alpha \left[\frac{1}{\vec{q}^{\,2}} - \frac{1}{q_z^2}\right] \\ &\times \quad \left(\theta(z)\frac{\mu_{253}^2}{\mu_{253}^2 + \vec{q}^{\,2}} + \theta(-z)\frac{\mu_{154}^2}{\mu_{154}^2 + \vec{q}^{\,2}}\right) e^{-2m^2 \frac{|\vec{q}^2|^2}{q_z^2} t} \ . \end{aligned}$$



Harmonic oscillator potential II

- Similarly, the self-energy terms can also be expressed in terms of W,

$$\frac{\delta m_{i\,t}^2}{m}\psi(\vec{k}) \quad = \quad -\int \frac{d^3q}{(2\pi)^3} W(\vec{q})\psi(\vec{k}) \ .$$

- Self-energy terms and effective exchange of gluons combine together,

$$\frac{\delta m_{1\,t}^2}{2m}\psi(\vec{k}) + \frac{\delta m_{3\,t}^2}{2m}\psi(\vec{k}) + \int \frac{d^3q}{(2\pi)^3} W(\vec{q})\psi(\vec{k}-\vec{q})$$
$$= \quad \int \frac{d^3q}{(2\pi)^3} W(\vec{q}) \left[\psi(\vec{k}-\vec{q}) - \psi(\vec{k})\right] \ .$$

- Taylor expanding the wave function

$$\psi(\vec{k}-\vec{q}) \quad = \quad \psi(\vec{k}) - q_i \frac{\partial}{\partial k_i}\psi(\vec{k}) + \frac{1}{2} q_i q_j \frac{\partial^2}{\partial k_i \partial k_j}\psi(\vec{k}) + \dots$$

Harmonic oscillator potential III

- we find the net effect of effective self-energy and effective exchange to be harmonic oscillator force:

$$\int \frac{d^3q}{(2\pi)^3} W(\vec{q}) \frac{1}{2} (q_i)^2 \frac{\partial^2}{\partial k_i^2} \psi(\vec{k}) \ .$$

- Finally,

$$\left[\frac{\vec{k}^2}{m} - B \right] \psi(\vec{k}) \quad + \quad \int \frac{d^3q}{(2\pi)^3} V_{C,\,BF}(\vec{q}) \, \psi(\vec{k} - \vec{q}) \\ - \quad \frac{4}{3} \frac{\alpha}{2\pi} b^{-3} \sum_i \tau_i \frac{\partial^2}{\partial k_i^2} \psi(\vec{k}) = 0 \ .$$

- The frequencies of harmonic oscillators in transverse and longitudinal directions should be the same to recover rotational invariance. This can also be achieved using appropriate gluon mass ansatz function.

Harmonic oscillator potential IV

- In particular, almost constant and large enough μ^2 will give $\tau_1 = \tau_2 = \tau_3 = \sqrt{\pi}/4$.

$$\vec{\tau} = \int_0^1 dv\, v(1-v^2) \begin{bmatrix} 1-v^2 \\ 1-v^2 \\ 2v^2 \end{bmatrix} \tau(v)$$

$$\tau(v) = \int_0^\infty du\ u^2\ e^{-u^2} \left[1 + \frac{u^2\ v^2}{2\ (m\,s)^2\ (\mu\,s)^2}\right]^{-1} ,$$

- Therefore, the final result could be written in the following way

$$\left[\frac{\vec{k}^{\,2}}{m} - \frac{1}{2}\tilde{\kappa}\,\Delta_{\vec{k}} - B\right]\psi(\vec{k}) + \int \frac{d^3q}{(2\pi)^3} V_{C,\,BF}(\vec{q})\,\psi(\vec{k}-\vec{q}) = 0\ ,$$

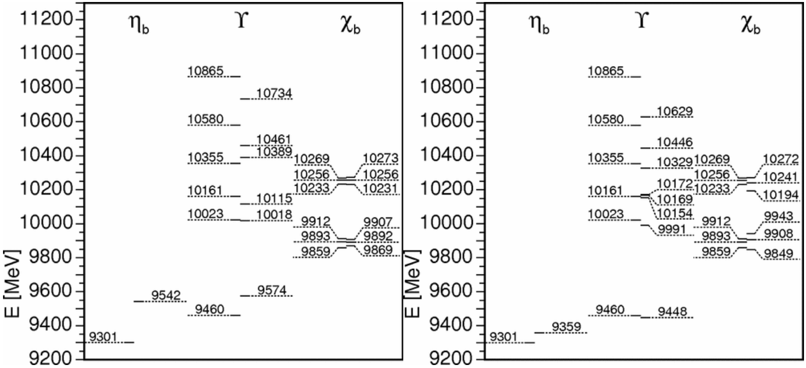
$$\tilde{\kappa} = \frac{m}{2}\omega^2 = \frac{\alpha}{(ms^2)^3}\frac{1}{36\sqrt{2\pi}}\ .$$

Phenomenological fit of parameters I

- Below are the results of numerical calculation obtained by S. Głazek and J. Młynik using the old generator of RGPEP and old nonrelativistic variables. (Phys. Rev. D 74, 105015 (2006))

	Experiment [31]	Fit to middle	Fit to all
λ [MeV]	⋯	3779.8	3252.3
m [MeV]	⋯	4835.9	4979.7
α	⋯	0.288 39	0.507 38
Υ10 580	10 580 ± 3.5	10 734	10 629
Υ^3D_1	⋯	10 461	10 446
Υ3S	10 355.2 ± 0.5	10 389	10 329
$\chi_2$2P	10 268.5 ± 0.72	10 273	10 272
$\chi_1$2P	10 255.5 ± 0.72	10 256	10 241
$\chi_0$2P	10 232.5 ± 0.9	10 231	10 194
Υ^1D_2	10 161.1? ± 2.2	⋯	10 172
Υ^3D_2	10 161.1? ± 2.2	⋯	10 169
Υ^1D_1	⋯	10 115	10 154
Υ2S	10 023.3 ± 0.31	10018	9991
$\chi_2$1P	9912.21 ± 0.57	9907	9943
$\chi_1$1P	9892.78 ± 0.57	9892	9908
$\chi_0$1P	9859.44 ± 0.73	9869	9849
Υ1S	9460.3 ± 0.26	9574	9448
η_b1S	9300.6 ± 10	9542	9359

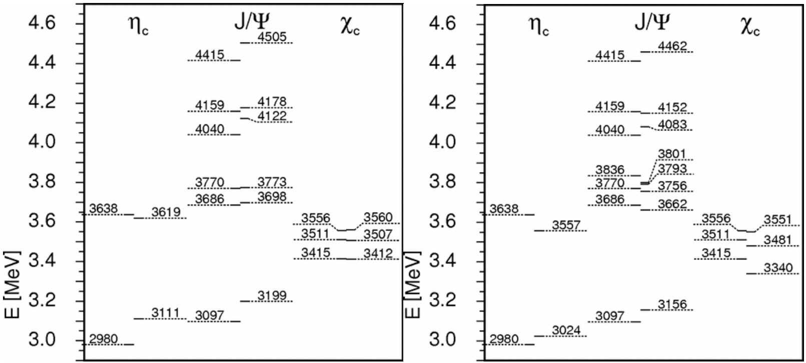
Phenomenological fit of parameters II



Phenomenological fit of parameters III

Meson	Experiment [31]	Fit to middle	Fit to all
λ [MeV]	⋯	1990.0	1934.2
m [MeV]	⋯	1553.3	1577.4
α	⋯	0.343 35	0.414 43
ψ4415	4415 ± 6	4505	4462
ψ4159	4159 ± 20	4178	4152
ψ4040	4040 ± 10	4122	4083
$^1D_2$	$3836? \pm 13$	⋯	3801
$^3D_2$	$3836? \pm 13$	⋯	3793
ψ3770	3770 ± 2.4	3773	3756
ψ2S	3686.093 ± 0.034	3698	3662
η_c2S	3638 ± 5	3619	3557
$\chi_2$1P	3556.26 ± 0.11	3560	3551
$\chi_1$1P	3510.59 ± 0.1	3507	3481
$\chi_0$1P	3415.16 ± 0.35	3412	3340
J/ψ1S	3096.916 ± 0.011	3199	3156
η_c1S	2980.4 ± 1.2	3111	3024

Phenomenological fit of parameters IV



Summary

- RGPEP together with gluon mass ansatz gives Coulomb force plus harmonic oscillator force between quarks.
- Without gluon mass ansatz one is left with only Coulomb potential (like QED, except of color factor 4/3).
- Energy of a single quark is infinite.
- In the $Q\bar{Q}$ system the harmonic oscillator force arises as an effect of cancellation between effective mass terms of quarks and antiquark, and an exchange of effective gluon.
- The resultant harmonic oscillator is the same as obtained earlier using different RGPEP generator and different non-relativistic momentum variables.

Outlook

- Analysis of Breit-Fermi terms in the new non-relativistic variables.
- Numerical calculation including different masses of quarks.
- However, the most desired course of study is 4^{th}-order calculation.
  - Will give precise form of spin dependent interactions.
  - Gluon mass ansatz will be replaced by true QCD effects.
- New generator allows for nonperturbative calculations, e.g., of gluon mass.
- Also, harmonic oscillator in baryons.