On (non-)polynomial f(R) gravity in the FRG

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Motivation

Asymptotic Safety Scenario for Gravity

- Einstein-Hilbert gravity perturbatively non-renormalisable
- With term $\propto$ Curvature^2 pert. renormalisable but not unitary
- Non-pert. renormalisable if
  an ultraviolet (UV) fixed point (FP) exists and
  number of relevant couplings is finite:

  Asymptotic Safety Scenario

  [Weinberg 1976, . . . , Reuter 1996, . . .]

  For details and more references see, *e.g.*, R. Percacci, *An introduction to covariant quantum gravity and asymptotic safety*, World Scientific 2017
- Convincing evidence for gravity . . .
- . . . but what if matter (Standard Model + ?Dark Matter?) is added?

Matter matters [Eichhorn, Percacci, . . .]
or gravity rules [Pawlowski, . . .] ???

Motivation

Introduction to $f(R)$ gravity for particle physicists

- Einstein Gravity: metric field $g_{\mu\nu}(x)$

  curvature (Riemann) tensor $R_{\mu\nu\rho\sigma} \propto [D_\mu, D_\nu]$

  (compare to Yang-Mills field strength tensor $F^{ab}_{\mu\nu} \propto [D_\mu, D_\nu]$)

  Ricci tensor $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$

  (Note: Different than in Yang-Mills theory one can contract spacetime and "representation" index!)

  curvature scalar $R = R^{\mu}{}_{\mu}$ (diffeomorphism invariant)

  (Lowest-order gauge invariant quantity $\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} = F^{ab}_{\mu\nu} F^{\mu\nu ba}$)

- Einstein-Hilbert action (Euclidean signature):

$$S_{EH} = \int d^4x \sqrt{\det g_{\mu\nu}} \left(\frac{\Lambda}{8\pi G} - \frac{\mathbf{R}}{16\pi G} \right) + S_{\mathrm{matter}}$$

  Asymptotically safe without matter in this truncation!

- . . . but if more terms generated by the RG are kept?

  . . . if matter is added?

Motivation

Introduction to $f(R)$ gravity for particle physicists

- As the Ricci scalar R is diffeomorphism invariant every function of it will be!
- $$S = \int d^d x \sqrt{g}\, f(R)$$

  is a diffeomorphism invariant action,

  called $f(R)$ gravity action.
- Einstein-Hilbert action: Polynomial $f(R)$ gravity of order one.
- Phenomenological constraints for higher-order terms are weak.

  NB: The "modified-gravity" approach wants to alleviate the need for Dark Matter and Dark Energy by extending Einstein-Hilbert gravity, see *e.g.* `www.cost.eu/COST_Actions/ca/CA15117`.
- Polynomial $f(R)$ gravity vs. non-polynomial versions, *e.g.*, the action may contain terms with logarithms.

Motivation

Asymptotic safety for $f(R)$ gravity?

See, *e.g.*,
K. Falls, D. F. Litim, K. Nikolakopoulos, C. Rahmede, Phys. Rev. D **93** (2016) 104022 [arXiv:1410.4815 [hep-th]]; arXiv:1607.04962 [gr-qc];
S. Gonzalez-Martin, T. R. Morris, Z. H. Slade, Phys. Rev. D **95** (2017) 106010 [arXiv:1704.08873 [hep-th]];
and references therein.

- Instead of an UV FP: Fixed function in the UV!
- In addition: Only finitely many relevant perturbations, *i.e.*, finitely many negative eigenvalues (= positive critical exponents) of the stability matrix of the linearised flow equation required.

Motivation

$f(R)$ gravity on spheres as background

- On maximally symmetric spaces
  (Spheres for $R > 0$ and hyperbolic spaces for $R < 0$,
  resp., deSitter and anti-deSitter for Lorentzian signature)
  $f(R)$ gravity includes all possible terms
  as, *e.g.*, then $R_{\mu\nu}R^{\mu\nu} \propto R^2$.
- Eigenvalues of Laplacian and Dirac operator are known for spheres S^d and hyperbolic spaces H^d.
  Note, however, different topology of S^d and H^d.
- The traces on the r.h.s. of the Wetterich equation can be performed by explicitly summing over eigenvalues.

In this talk:

Couple matter (scalar, fermion and vector fields) to $f(R)$ gravity, and determine UV Fixed Functions by employing the Wetterich eq. with spheres as background.

RG equation for $f(R)$ gravity

N. Ohta, R. Percacci, G. P. Vacca, Eur. Phys. J. C **76** (2016) 46 [arXiv:1511.09393 [hep-th]].

Hessian for the exponential parameterisation

- Effective action $\Gamma_k = \int d^dx \sqrt{g}\, f(R)$ with coupling constants $g_0(k), g_1(k), g_2(k)\ldots$, one for every power of R
- Exponential split $g_{\mu\nu} = \bar{g}_{\mu\rho}(e^h)^\rho{}_\nu$
- York decomposition (good spin!)
- Gauge fixing $S_{GF} = \frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}F_\mu F_\nu$ with $F_\mu = D_\rho h^\rho{}_\mu - \frac{\beta+1}{d}D_\mu h$ and $\beta\to\infty$, then $\alpha\to 0$.
- $\Gamma^{(2)}_{grav} = -\frac{1}{4}f'(\bar{R})h^{TT}_{\mu\nu}\left(\Delta + \frac{2}{d(d-1)}\bar{R}\right)h^{TT\,\mu\nu} + \ldots$ Laplacian $\Delta = -\bar{D}^\mu\bar{D}_\mu$ depends on background curvature $\bar{R}$ and on spin.

RG equation for $f(R)$ gravity

Eigenvalues and Regulators

- The eigenvalues of the Laplacian on the sphere are $\lambda_\ell^{s=m} = \bar{R}\frac{\left(\ell(\ell+d-1)-m\right)}{d(d-1)}$ $, \ell = m, m+1, \ldots$ with $m = 0, 1, 2$ for scalar, transverse vector and transverse-traceless tensor fields.
- Regulators chosen to depend on endomorphisms linear in $\bar{R}$: $R_k^{S,V,T}(\Delta - \alpha^G_{S,V,T}\bar{R})$
- $\mathrm{Tr}_{(s)}\,\mathcal{O} = \sum_{\ell=\ell_{min}}^{\infty} M_d^s(\ell)\lambda_\ell^{\mathcal{O},s}$ with $\ell_{min} = 2$ in gravity sector

---

$$\begin{aligned}
\partial_t \Gamma_k|_{grav} = {} & \frac{1}{2}\,\mathrm{Tr}_{(2)}\left(\frac{(\partial_t f'(\bar{R}))R_k^T + f'(\bar{R})\partial_t R_k^T}{f'(\bar{R})(P_k^T + \alpha_T \bar{R} + \frac{2}{d(d-1)}\bar{R})}\right) \\
& -\frac{1}{2}\,\mathrm{Tr}'_{(1)}\left(\frac{\partial_t R_k^V}{P_k^V + \alpha_V^G \bar{R} - \frac{1}{d}\bar{R}}\right) \\
& +\frac{1}{2}\,\mathrm{Tr}''_{(0)}\left(\frac{(\partial_t f''(\bar{R}))R_k^S + f''(\bar{R})\partial_t R_k^S}{f''(\bar{R})(P_k^S + \alpha_S^G \bar{R} - \frac{1}{d-1}\bar{R}) + \frac{d-2}{2(d-1)} f'(\bar{R})}\right)
\end{aligned}$$

Matter Part of the RG equation

NA & F. Saueressig, to be published

Adding matter

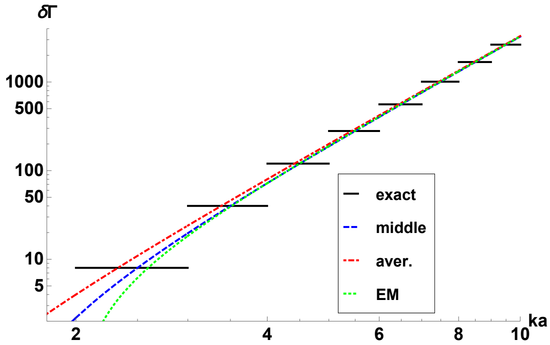
- N_S scalar fields, regulator parameter α_S^M
- N_D Dirac fields, regulator parameter α_D
- N_V vector fields, regulator parameters α_V^M and α_S^M for transverse vectors and long. vectors & ghosts, resp.
- Standard Model: $N_S = 4$ (Higgs),
  $N_D = 22.5$ (24 with r.h. ν),
  $N_V = 12$

Matter Part of the RG equation

Performing the traces

- Seven different traces, here fermions as example.
- Flat Litim-type regulator $R_k^D(z) = (k^2 - z)\Theta(k^2 - z)$
- Direct summation provides a staircase function smoothened with three different approximations

Matter Part of the RG equation

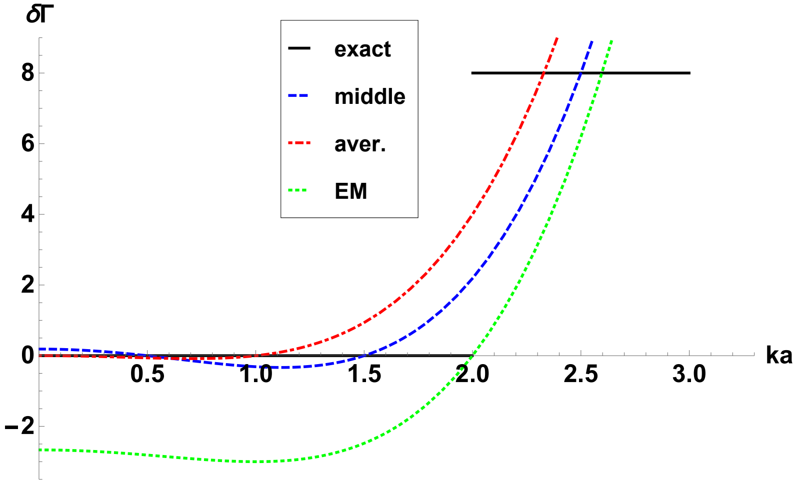


Smoothening

1. middle of the staircase
2. average upper & lower
3. Euler-Maclaurin & integral only

k: RG scale $a = \sqrt{\frac{d(d-1)}{\bar{R}}}$ radius of sphere

Matter Part of the RG equation



Very different at small RG scales!

NB: Plot is for $\alpha_D = 0$ type-II-regulator

Matter Part of the RG equation

$$\partial_t \Gamma_k|_D = -N_D \frac{V(d)k^d}{\Gamma(d/2+1)(4\pi)^{d/2}} \frac{k^2+\frac{d}{2}\alpha_D \bar{R}}{k^2+\alpha_D \bar{R}} 2^{\lfloor d/2 \rfloor} \left(1 - \mathrm{A}\frac{\bar{R}}{k^2} + \mathcal{O}(\bar{R}^2)\right),$$

Linear term in $\bar{R}$ in

middle appr.:	A =	(d+1)/24
averag. appr.:	A =	(d-2)/24
EM appr.:	A =	d/24

In $d = 4$: $\partial_t \Gamma_k|_D = -N_D \frac{2V(4)k^4}{(4\pi)^2}$

$$\frac{k^2}{k^2+\alpha_D \bar{R}} \left(1 + (\alpha_D - 3/16)\frac{\bar{R}}{k^2}\right)\left(1 + (\alpha_D - 1/48)\frac{\bar{R}}{k^2}\right) \quad \text{middle appr.}$$

$$\left(1 + (\alpha_D - 1/12)\frac{\bar{R}}{k^2}\right) \quad \text{averag. appr.}$$

$$\frac{k^2}{k^2+\alpha_D \bar{R}} \left(1 + (\alpha_D - 1/3)\frac{\bar{R}}{k^2}\right)\left(1 + (\alpha_D + 1/6)\frac{\bar{R}}{k^2}\right) \quad \text{EM appr.}$$

Type-I ($\alpha_D = 1/4$) and Type-II ($\alpha_D = 0$) regulators lead to different behaviour: Fermions destabilise / stabilise the UV FP !?!

Matter Part of the RG equation

Type-I ($\alpha_D = 1/4$) and Type-II ($\alpha_D = 0$) regulators lead to different behaviour: Fermions destabilise / stabilise the UV FP !?!

Positivity of argument or regulator function: $\alpha_D \leq d/4(d-1)$

No additional pole in flow: $\alpha_D \geq 0$

Both type of regulators are allowed! But provide different physics.

Dimensionless variables

- $\rho = \bar{R}/k^2$ and $\varphi(\rho) = f(\bar{R})/k^d$
- $\partial_t \Gamma_k = \int d^d x \sqrt{g} \partial_t f(\bar{R}) = V(d) k^d \left(\partial_t \varphi(\rho) + d\varphi(\rho) - 2\rho\varphi'(\rho)\right)$
- Notation: UV Fixed Function $\phi(\rho)$

RG equation in four dimensions

Flow equation in averaging approximation

- Flow eq. consists of seven terms:
  Tensor, vector (gravity) ghost and conformal scalar mode in gravity sector;
  Scalar, fermion, scalar (gauge) ghost and vector in matter sector.
- $\partial_t \varphi(\rho) + 4\varphi(\rho) - 2\rho\varphi'(\rho) = \frac{T_1(\partial_t\varphi' + 2\varphi' - 2\rho\varphi'')}{\varphi'} + \ldots - N_D D$
  with
  $T_1 = \frac{1}{(4\pi)^2}\,\frac{5}{12}\,(1 + (\alpha_T - 1/6)\rho)(1 + (\alpha_T - 2/3)\rho)$
  ...
  $D = \frac{1}{(4\pi)^2}\,2\,(1 + (\alpha_D - 1/12)\rho)$
- Assuming $\varphi(\rho)$, resp., $\phi(\rho)$ to be a polynomial of degree N:
  $\phi(\rho) = \frac{1}{(4\pi)^2}\,\sum_{n=0}^{N} g_n^\star \rho^n$

RG equation in four dimensions

Influence of matter on the cosmological constant

- $f(R) = \frac{\Lambda_k}{8\pi G_k} - \frac{R}{16\pi G_k} + \mathcal{O}(R^2)$
- $\Lambda_k = -\frac{g_0}{2g_1}k^2 \qquad \text{and} \qquad G_k = -\frac{\pi}{k^2 g_1}$
- $g_0^\star = \frac{11}{24} + \frac{1}{8}\frac{1}{1+g_1^\star/6g_2^\star} + \frac{1}{8}(N_S - 4N_D + 2N_V)$
  resp.
  $g_0^{complete} \approx g_0^{gravity} + \frac{1}{8}(N_S - 4N_D + 2N_V)$ with
  $g_0^{gravity} \approx 1/2$
- Every bosonic degree of freedom contributes $+1/8$, every fermionic degree of freedom $-1/8$ to the constant g_0: Fixed point value changed accordingly.
- Standard Model matter content is dominated by fermions: negative value for $g_0^\star$ and thus for the fixed point value of the cosmological constant $\Lambda^\star$

RG equation in four dimensions

Differential equation for Fixed Function

Third order differential equation with

1. Fixed singularity at $\rho = 0$ (always for physical reason)
2. Physically acceptable solution: $G > 0 \Rightarrow g_1 < 0$
$\Rightarrow \phi(\rho)$ possesses minimum
$\Rightarrow$ term $\propto 1/\phi'(\rho)$ produces singularity
3. Maximally one more singularity allowed

NB: Several cases studies
especially those where solution for pure gravity is given in Ohta/Percacci/Vacca.
Here: Type I and Type II regulators

RG equation in four dimensions

NA & F. Saueressig, to be published [arXiv:17mm.nnnnn]

Polynomial approximations for fixed functions in UV

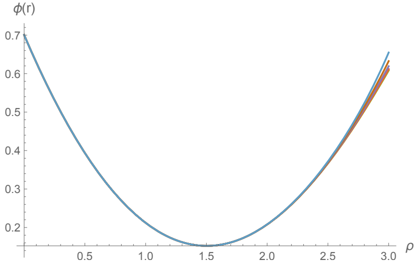
$\rho \to 0$ corresponds to $k^2 \to \infty$ for fixed $\bar{R}$:

Polynomial approximations around $\rho = 0$ up to order 14

Type I regulator: $\alpha_T = \alpha_V^{G,M} = \alpha_S^{G,M} = 0$ and $\alpha_D = 1/4$

Pure gravity:

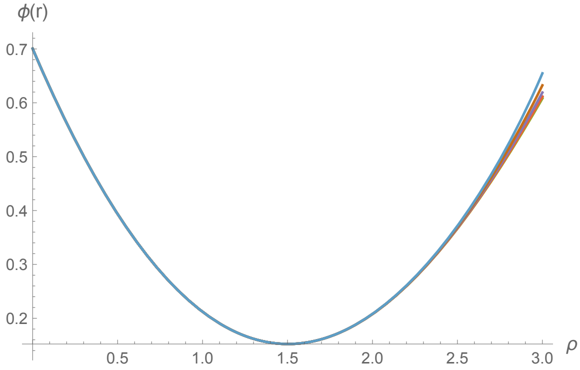
$$(4\pi)^2\phi(\rho) \approx 0.700598 - 0.74115\rho + 0.255198\rho^2 - 0.00180589\rho^3 - \ldots$$



At $\rho = 3/2$ cancelation s.t. zero of T_1 removes potential singularity $\phi'(\rho) = 0$

RG equation in four dimensions

Pure gravity:



All determined critical exponents real, only two positive:
$\Theta_0 = 4$ (also analytically), $\Theta_1 \approx 2.11$, $\Theta_2 \approx -4.5$, etc.

RG equation in four dimensions

Matter contribution to ODE for $\phi(\rho)$:

$$\frac{1}{2}(N_S - 4N_D + 2N_V)$$
$$+\frac{1}{12}(N_S - 4N_D + \frac{3}{2}N_V)\,\rho$$
$$+\frac{1}{36}(-N_S + 11N_V)\,\rho^2 + \mathcal{O}(\rho^3).$$

(N_S, N_D, N_V)	(1,0,0)	(4,0,0)	(0,0,1)	(0,0,2)
$g_0^\star$	0.818143	1.17375	1.07152	1.48687
$g_1^\star$	-0.691112	-0.53307	-0.683569	-0.51518
Θ_1	2.1	1.9	3.3	3.3

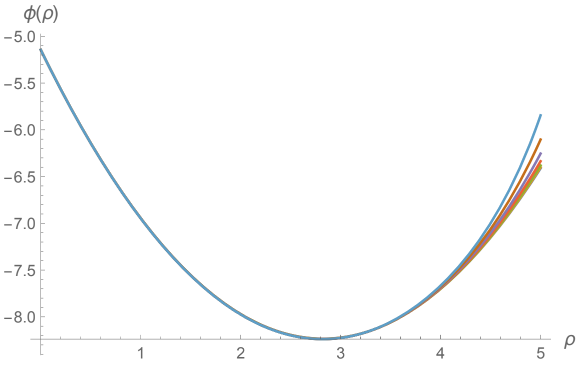
(N_S, N_D, N_V)	(0,3,0)	(0,6,0)
$g_0^\star$	-0.799725	-2.30002
$g_1^\star$	-1.24201	-1.74274
Θ_1	2.2	2.2

Bosons increase, fermions decr. $g_1^\star$, $g_1^\star \propto -1/G^\star \Rightarrow$ Bosons increase, fermions decr. $G^\star$

RG equation in four dimensions

Standard Model matter content $N_S = 4$, $N_D = 22.5$ and $N_V = 12$:

$(4\pi)^2\phi(\rho) \approx$
$-5.14542 - 2.19065\rho + 0.387687\rho^2 - 0.0000917095\rho^3 - \ldots$

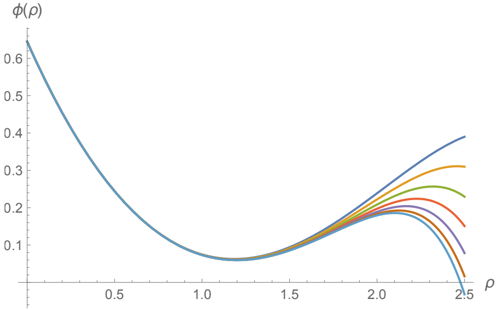


$\Theta_0 = 4$, $\Theta_1 \approx 2$, $\Theta_{2,3} \approx -27 \pm 31i$, etc.

RG equation in four dimensions

Polynomial approximations for fixed functions in UV

Type II regulator: $\alpha_T = \alpha_V^{G,M} = \alpha_S^{G,M} = 0$ and $\alpha_D = 1/4$
Pure gravity:
$(4\pi)^2\phi(\rho) \approx 0.644905 - 1.05682\rho + 0.53372\rho^2 - 0.0470553\rho^3 - \ldots$



RG equation in four dimensions

Matter contribution to ODE for $\phi(\rho)$:

$$\frac{1}{2}(N_S - 4N_D + 2N_V)$$
$$+\frac{1}{12}(N_S + 2N_D - 3N_V)\,\rho$$
$$+\frac{1}{36}(-N_S + 11N_V)\,\rho^2 + \mathcal{O}(\rho^3).$$

Add only scalars:

N_S	0	3	6	9	11
$g_0^\star$	0.644905	1.01172	1.3788	1.74694	1.99222
$g_1^\star$	-1.05682	-0.878279	-0.688708	-0.481227	-0.319026
$g_2^\star$	0.53372	0.489098	0.430368	0.339882	0.249306
Θ_1	1.76	1.63	1.46	≈ 1.2	≈ 0.9

Similar to type-I regulator.

RG equation in four dimensions

Matter contribution to ODE for $\phi(\rho)$:

$$\tfrac{1}{2}(N_S - 4N_D + 2N_V)$$
$$+\tfrac{1}{12}(N_S + 2N_D - 3N_V)\,\rho$$
$$+\tfrac{1}{36}(-N_S + 11N_V)\,\rho^2 + \mathcal{O}(\rho^3).$$

Add only fermions:

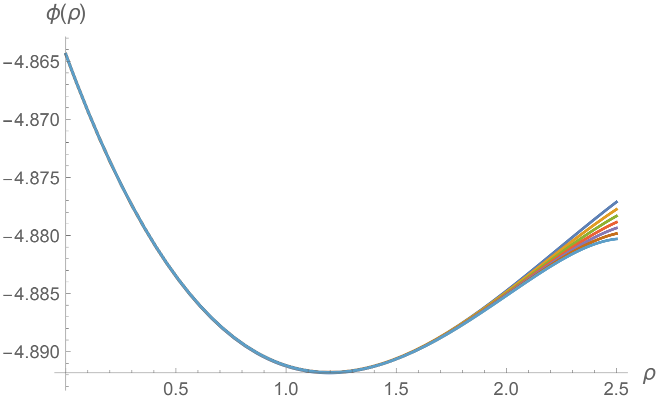
N_D	0	3	6	9
$g_0^\star$	0.644905	-0.856389	-2.35813	-3.86084
$g_1^\star$	-1.05682	-0.794662	-0.528304	-0.252755
Θ_1	1.76	1.73	1.71	1.79

N_D	10	11	12	15
$g_0^\star$	-4.36225	-4.86439	-5.43094	-6.8698
$g_1^\star$	-0.155924	-0.0517027	0.7863	0.332336
Θ_1	2.05	6.0	>1000	>1000

Opposite reaction as for type-I regulators, at $N_D \approx 11.5$ Newton's constant changes sign and solution becomes unphysical.

RG equation in four dimensions

Eleven fermions, shortly before solution "tilts":



Note the change of scale, the fixed function for $N_D = 11$ is varying less by one percent in the shown interval.

RG equation in four dimensions

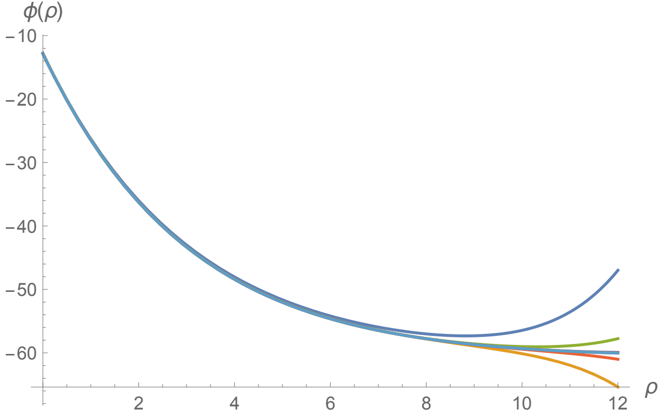
Matter contribution to ODE for $\phi(\rho)$:

$$\frac{1}{2}(N_S - 4N_D + 2N_V)$$
$$+\frac{1}{12}(N_S + 2N_D - 3N_V)\,\rho$$
$$+\frac{1}{36}(-N_S + 11N_V)\,\rho^2 + \mathcal{O}(\rho^3).$$

PRELIMINARY:
Adding only vector fields might even NOT be possible!!!
No solution found until now . . .

RG equation in four dimensions

$\phi(\rho)$ for Standard Model content:



To achieve a solution for the matter content of the SM:
First, $N_S = 4$, $N_D = 22.5$, $N_V = 3$ & then increase # of vectors.
Solutions are quite distinct from all the ones discussed before!!

RG equation in four dimensions

$\phi(\rho)$ for Standard Model content:

- $g_0^\star$ is not only negative but also quite large in modulus.
- $g_1^\star$ is also quite large implying small FP value for Newton's constant.
- Not clear whether $\phi(\rho)$ possesses a minimum.
- Critical exponents do not show convergence (until polynomial order $N = 9$)

PRELIMINARY! Search for better solution on-going . . .

RG equation in four dimensions

NA & F. Saueressig, in preparation

Global Fixed Functions: Global quadratic solutions

- Pure Gravity: Verified the five solutions given by Ohta/Percacci/Vacca. But unstable under variation of parameters.
- With matter: Quite a number of solutions, *e.g.*, for $\alpha_T = -53/94, \alpha_V^G = -83/564, \alpha_S^G = -3/47$ and $\alpha_D = {}^1/_{12}, N_S = N_V = 0$:

$$\phi(\rho) = g_0^\star + g_1^\star \rho + g_1^\star \rho^2$$

with
$g_0^\star = 89/72 - N_D/2$, $g_1^\star = -101/94$, and $g_2^\star = 1414/6627$
is GLOBALLY a solution.

RG equation in four dimensions

Global Fixed Functions: Numerical solution

- Solve from $\rho = 0$ to minimimum and require a zero of T_1 (cancel the singularity).
- Solve from minimum to 2nd fixed singularity.
- Match above 2nd singularity to semi-analytically determined asymptotic behaviour.
  E.g., for pure gravity:
  $(4\pi)^2\phi(\rho) = a_{symp}\rho^2 + \frac{1053\, a_{symp}}{50-864\, a_{symp}}\rho + \frac{4105728\, a_{symp}^2 + 6727320\, a_{symp} - 1943075}{384(25-432\, a_{symp})^2} + \mathcal{O}(1/\rho)$
- Repeat until solution and its derivatives become continueous, i.e.,employ numerical multi-shooting.

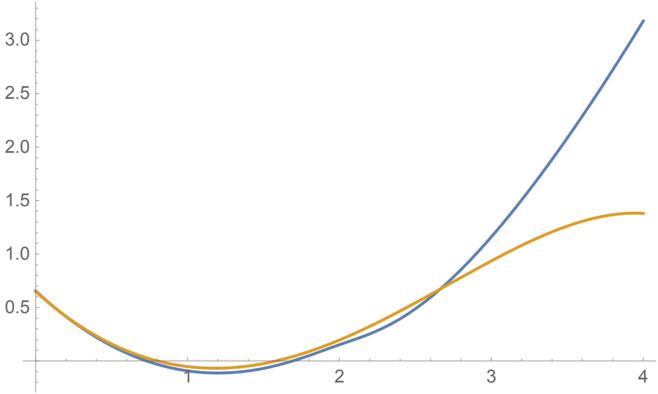
RG equation in four dimensions

Global Fixed Functions: Numerical solution

VERY PRELIMINARY!

E.g.:

Pure gravity with $\alpha_T = -1/6$, $\alpha_V^G = 1/2$, $\alpha_S^G = 1/3$



Blue: Numerical solution **Orange: Polynomial approximation**

RG equation in four dimensions

With matter:

- Solution (if it exists) for small ρ captured very well by polynomial approximation
  (at least, for until now investigated cases).
- Leading asymptotic behaviour $\propto \rho^2$ not changed by matter, subleading changed.

Conclusions

$f(R)$ gravity with matter on spheres as background

Asymptotic safety scenario verified and fixed functions for a

- FRG study of $f(R)$ gravity coupled to scalar, Dirac fermion and Maxwell vector fields
- exponential parameterization of the metric
- traces performed explicitely by summing over eigenvalues
- dependence on regulator parameters, *e.g.*, type-I vs type-II regulators
- Solutions with matter content up to SM:
  - polynomial approximations in UV (study almost completed)
  - global solutions (ongoing investigation)
- Generically: Two relevant directions
  - scaling exponent for g_0 (**cosmological constant**): $\Theta_0 = 4$
  - for a linear combination of g_1 and g_2: $\Theta \approx 1 \ldots 3$
    Superposition of linear and quadratic term is relevant!