Flucton in quantum mechanics: two-loop correction

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(collaboration with Adrian Escobar and Edward Shuryak)

The classical path: Flucton

For the density matrix given by the Euclidean time path integral with fixed coinciding endpoints, we calculate the classical trajectory, the classical action, Green function and then using Feynman diagrams the determinant and two-loop correction for the AHO, DWP, Sine-Gordon.

What is QM meaning of loop expansion?

Path integrals in Euclidean time

Path integrals in Euclidean time ($t = -i\,\tau$)

Density matrix in Quantum Mechanics

$$\rho(x_i,\, x_f;\, \tau_0) \;=\; \langle x_f | e^{-H\,\tau_0} | x_i \rangle = N \int_{x_i = x(-\tau_0/2)}^{x_f = x(\tau_0/2)} [dx]\, e^{-S}\,, \qquad (\hbar = 1)$$

H is Hamiltonian and N a normalization factor

The **action** S

$$S[x(\tau)] = \int_{-\frac{\tau_0}{2}}^{\frac{\tau_0}{2}} d\tau \left[\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right] > 0$$

$P(x_0;\, \tau_0) \equiv \rho(x_0,\, x_0;\, \tau_0)$ gives the **probability** $|\psi(x_0)|^2$ for the specific coordinate value x_0 to be found in the ensemble

Quantum statistical mechanics at temperature T

Density matrix as a sum over decreasing exponentials

$$P(x_0;\,\beta) \;=\; \langle x_0|e^{-H\,\beta}|x_0\rangle \;=\; \sum_n |\psi_n(x_0)|^2 e^{-E_n\,\beta}$$

$(H\,|n\rangle = E_n\,|n\rangle)$, τ defined on a circle with circumference $\beta = \tau_0$ (the Matsubara time)

The limit $\beta \to \infty$ (low temperature limit $T = \hbar/\beta \to 0$)

$$P(x_0;\,\beta \to \infty) \;\sim\; |\psi_0(x_0)|^2$$

describes the ground state

Saddle-point method

How to evaluate $N \int [dx]\, e^{-S}$?

Take an **extremal path** $\delta S[X(\tau)] = 0$

$$\frac{d^2}{d\tau^2} X = +\frac{d}{dx} V(x)|_{x=X}$$

which exists(!) and calculate

Classical action, Quadratic fluctuations + loop-corrections

$$\int [dx]\, e^{-S} \sim \frac{\exp\left(-S[X(\tau)]\right)}{\sqrt{\mathrm{Det}\left(O_X\right)}} \times \left[1 + g\, B_1 + g^2\, B_2 + \ldots\right]$$

$$= \exp\left[-\left(S[X(\tau)] + \frac{1}{2}\mathrm{Det}(O_X) - g\, B_1 - g^2\, \tilde{B}_2 - \ldots\right)\right]$$

where B_1, B_2,...B_n are given by the sum of $2-, 3-, (n+1)$-loop Feynman diagrams

Flucton

Flucton

Semiclassical object: Flucton (Shuryak, 1988)

A flucton $X_{flucton}(\tau)$ is an extremal path, in Euclidean time, with zero energy $E = T_{kinetic} - V = 0$

$$\frac{d}{d\tau} X_{flucton}(\tau) \; = \; \sqrt{2\, V(X_{flucton})}$$

subject to the condition

$$X_{flucton}(0) = X_{flucton}(\beta) = x_0$$

Unlike instanton, flucton is **not** related with topology or symmetries

Harmonic oscillator

For the harmonic oscillator the flucton is

$$X_{flucton}(\tau) \;=\; x_0 \,\frac{\left(e^{\beta-\tau} \;+\; e^{\tau}\right)}{e^{\beta}+1}$$

defined for $\tau \in [0, \beta]$. The classical action gives the **exact result**

$$S[X_{flucton}(\tau)] \;=\; x_0^2 \;\tanh\left(\frac{\beta}{2}\right)$$

The particle distribution

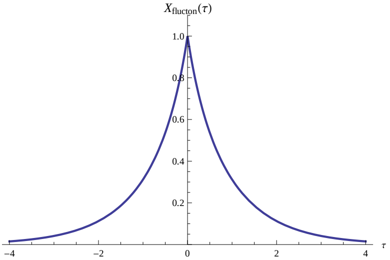
$$P(x_0;\,\beta) \propto \exp\left(-\frac{x_0^2}{\coth(\frac{\beta}{2})}\right)$$

is Gaussian at any temperature

Anharmonic oscillator $V = \frac{1}{2}x^2\,(1 + g\,x^2)$

At zero temperature the flucton is

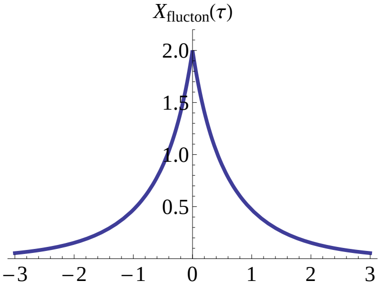
$$X_{flucton}(\tau) = \frac{\sqrt{g}\,x_0}{\cosh(\tau) + \sqrt{1 + g\,x_0^2}\,\sinh(\tau)}$$



$$S[X_{flucton}(\tau)] = \frac{2}{3}\,\frac{(1 + g\,x_0^2)^{\frac{3}{2}} - 1}{g}\,, \quad P(x_0;\,\beta \to \infty) \sim e^{-\frac{2}{3}\,\frac{(1+g\,x_0^2)^{\frac{3}{2}}-1}{g}}$$

Double Well Potential $V = \frac{1}{2}x^2\left(1-\sqrt{2}\,g\,x+g^2\,x^2\right)$

$$X_{flucton}(\tau) = \frac{x_0}{e^{|\tau|}(1+\sqrt{2}\,g\;x_0)-\sqrt{2}\,g\;x_0}$$



$$S[X_{flucton}] = x_0^2(1+\frac{2\sqrt{2}\,gx_0}{3})\ , \quad P(x_0;\,\beta\to\infty) \sim e^{\left(-x_0^2-\frac{2\sqrt{2}\,g}{3}x_0^3\right)}$$

Two loop-correction B_1

$$P(x;\, \beta \to \infty) \;=\; \exp\left[-S[X_{flucton}(\tau)] - \frac{1}{2}\mathrm{Det}(O_{X_{flucton}}) + g\, B_1 + \ldots\right]$$

$$\equiv\; e^{-2\,\phi(x_0)}$$

Green function for AHO

The Green function of operator O_X, $G(\tau_1, \tau_2; X)$

$$\frac{\mathrm{Sech}\left[\frac{1}{2}\left(-|\tau_-|+\tau_+\right)\right]\left(X\mathrm{Cosh}\left[\frac{1}{2}\left(|\tau_-|+\tau_+\right)\right]+\mathrm{Sinh}\left[\frac{1}{2}\left(|\tau_-|+\tau_+\right)\right]\right)}{4X\left(\mathrm{Cosh}\left[\frac{1}{2}\left(|\tau_-|+\tau_+\right)\right]+X\mathrm{Sinh}\left[\frac{1}{2}\left(|\tau_-|+\tau_+\right)\right]\right)^2\left(1+X\mathrm{Tanh}\left[\frac{1}{2}\left(-|\tau_-|+\tau_+\right)\right]\right)^2}$$

$$\times\Bigg[X\mathrm{Cosh}\left[|\tau_-|-\tau_+\right]\left(1+3X^2+X\left(3+X^2\right)\mathrm{Tanh}\left[\frac{1}{2}\left(-|\tau_-|+\tau_+\right)\right]\right)$$

$$\left(-1+X^2\right)\left(-4+5X^2-3X\left(-|\tau_-|+\tau_+\right)\right)\mathrm{Tanh}\left[\frac{1}{2}\left(-|\tau_-|+\tau_+\right)\right]$$

$$-X\left(1+3X\left(X+\left(-1+X^2\right)\left(-|\tau_-|+\tau_+\right)\right)\right)\Bigg]$$

$$X \equiv \sqrt{1+g\,x_0^2}$$

$\tau_- = \tau_2 - \tau_1, \quad \tau_+ = \tau_1 + \tau_2$

Green function for DWP

The Green function of operator O_X

$$G(\tau_1, \tau_2; X) = \frac{e^{-|\tau_1-\tau_2|}}{2\left(e^{\tau_1}(1+X)-X\right)^2\left(e^{\tau_2}(1+X)-X\right)^2}$$

$$\Big[8\, e^{\frac{1}{2}(\tau_1+\tau_2+3\,|\tau_1-\tau_2|)}\, X^3\,(1+X) - 8\, e^{\frac{1}{2}(3\tau_1+3\tau_2+|\tau_1-\tau_2|)}\, X\,(1+X)^3$$

$$+\, e^{2\,(\tau_1+\tau_2)}\,(1+X)^4 - 6\, e^{(\tau_1+\tau_2+|\tau_1-\tau_2|)}\, X^2\,(1+X)^2\,|\tau_1-\tau_2|$$

$$+\, e^{(\tau_1+\tau_2+|\tau_1-\tau_2|)}\left(6\,X^4\,(\tau_1+\tau_2)+12X^3(1+\tau_1+\tau_2)+6X^2(3+\tau_1+\tau_2)+4X-1\right)$$

$$-e^{2\,|\tau_1-\tau_2|}\,X^4\,\Big] \qquad (\tau_1,\,\tau_2>0)$$

where

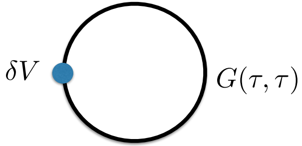
$$X \equiv x_0\,\sqrt{2}\,g$$

Determinant in flucton background

Relating the determinant and the Green function (Brown and Creamer,1978)

$$\frac{\partial \log \mathrm{Det}\,(O_X)}{\partial X} = \int d\tau\, G(\tau,\tau;X)\frac{\partial V_{flucton}(\tau)}{\partial X}$$

$V_{flucton} \equiv V(X_{flucton})$. Symbolic one-loop diagram



Determinant

For AHO

$$\log \text{Det}(O_x) = 2 \log[X(1+X)/2]$$

$$X \equiv \sqrt{1+g\,x_0^2}$$

For DWP

$$\log \text{Det}(O_x) = 4 \log[1+X]$$

$$X \equiv \sqrt{2}\,g\,x_0$$

Feynman´s rules: AHO

- nth-loop contribution = sum over all closed *n*-loop diagrams
- $v_3(\tau; X) = \frac{12\sqrt{g}\sqrt{-1+X^2}}{\text{Cosh}[\tau]+X\text{Sinh}[\tau]}$ and $v_4 = 12\,g$ provide triple and quartic vertices
- vertices connected by "propagators" $G(\tau_1, \tau_2; X) = \langle x(\tau_1)\, x(\tau_2)\rangle$
- integrals over time for each vertex should be performed, symmetry factors
- Normalization by subtracting the same diagrams with the *vacuum vertice* $v_{4,0} = 12\,g$ and the *vacuum propagator*

$$G_0 = G(\tau_1, \tau_2; X)\,|_{X\to 0} = \frac{e^{-|\tau_1-\tau_2|}}{2} - \frac{e^{-\tau_1-\tau_2}}{2}$$

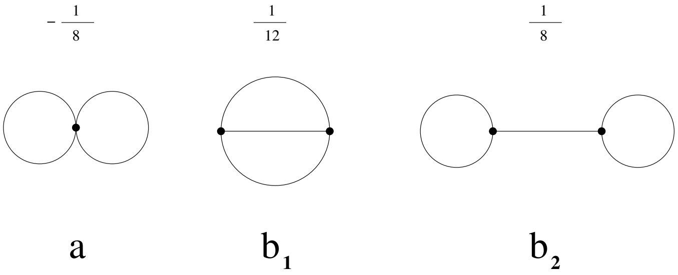
Feynman´s rules: DWP

- nth-loop contribution = sum over all closed *n*-loop diagrams
- $v_3(\tau; X) = \frac{6\sqrt{2}\, g(X+e^{\tau}(1+X))}{-X+e^{\tau}(1+X)}$ and $v_4 = 24\, g^2$ provide triple and quartic vertices
- vertices connected by "propagators" $G(\tau_1,\, \tau_2; X) = \langle x(\tau_1)\, x(\tau_2)\rangle$
- integrals over time for each vertex should be performed, symmetry factors
- Normalization by subtracting the same diagrams with *vacuum vertices* $v_{3,0} = 6\sqrt{2}\, g$, $v_{4,0} = 24\, g^2$ and the *vacuum propagator*

$$G_0 = G(\tau_1,\, \tau_2; X)\, |_{X\to 0} = \frac{e^{-|\tau_1-\tau_2|}}{2} - \frac{e^{-\tau_1-\tau_2}}{2}$$

The two-loop correction

The two-loop correction B_1 is the sum of three two-loop diagrams



Anharmonic oscillator

$$a = -\frac{1}{8}\, v_4 \int_0^\infty [G^2(\tau,\tau;X) - G_0^2(\tau,\tau)]d\tau$$

$$a = \frac{3\left(-32 - 96X - 29X^2 + 74X^3 + 35X^4\right)}{1120X^2(1+X)^2}$$

$$- \frac{9\left(2(2-3X^2)\mathrm{Log}2 - 2\left(2-3X^2\right)\mathrm{Log}(1+X) - 3X\left(1-X^2\right)\mathrm{PL}\left[2, \frac{-1+X}{1+X}\right]\right)}{140X\left(1-X^2\right)}$$

$\mathrm{PL}[n,z] = \sum_{k=1}^{\infty} z^k/k^n$ is the polylogarithm function

Irrational contributions occur coming from Log2 , $\mathrm{Log}(1+X)$ and PolyLog $\left[2, \frac{X}{1+X}\right]$

$$b_1 = \frac{1}{12}\int_0^\infty \int_0^\infty \mathrm{v}_3(\tau_1; X)\,\mathrm{v}_3\,(\tau_2; X)\,G^3(\tau_1,\, \tau_2; X)\,d\tau_1\, d\tau_2$$

$$b_1 = \frac{-140 - 184X - 272X^2 - 193X^3 + 478X^4 + 455X^5}{1680X^3(1+X)^2}$$

$$-\frac{3\left(2(2-3X^2)\mathrm{Log}2 - 2\left(2-3X^2\right)\mathrm{Log}[1+X] - 3X\left(1-X^2\right)\mathrm{PL}\left[2, \frac{-1+X}{1+X}\right]\right)}{70X\left(-1+X^2\right)}$$

Irrational contributions occur again, in b_2 as well, however, eventually, after summing all three diagrams, the two-loop correction takes an amazingly simple form,

$$B_1^{(AHO)} = \frac{(1-X)\,(5+16X+25X^2+17X^3)}{12X^3(1+X)}$$

$$B_1^{(DWP)} = -\frac{X\,(4+3\,X)}{(1+X)^2}$$

All log *and PolyLog terms disappear!*

$$2\,\phi^{(AHO)} = \frac{2}{3}\frac{(1+g\,x^2)^{\frac{3}{2}}-1}{g} + \log[(1+g\,x^2+\sqrt{1+g\,x^2})/2]$$

$$+g\,\frac{(1-\sqrt{1+gx^2})\left(5+16\sqrt{1+gx^2}+25\,(1+gx^2)+17\,(1+gx^2)^{3/2}\right)}{12(1+gx^2)^{3/2}\left(1+\sqrt{1+gx^2}\right)}+\ldots$$

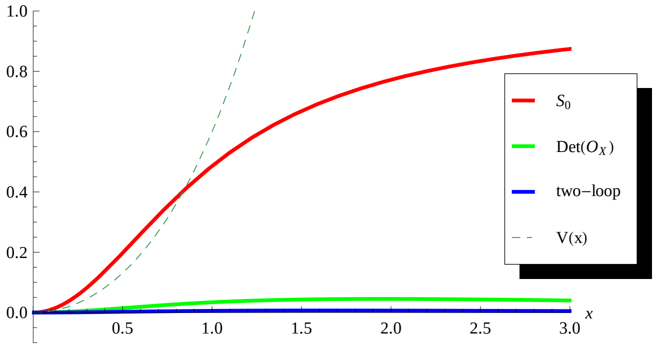
$$2\,\phi^{(DWP)} = x^2 + \frac{2\sqrt{2}\,g\,x^3}{3} + 2\,\log(1+\sqrt{2}\,g\,x) + g^2\,\frac{\sqrt{2}\,g\,x\,(4+3\sqrt{2}\,g\,x)}{(1+\sqrt{2}\,g\,x)^2}+\ldots$$

$$2\,\phi^{(Sine-Gordon)} = \frac{16}{g^2}\sin^2\left(\frac{g\,x}{4}\right) - 2\,\log[\cos\left(\frac{g\,x}{4}\right)] + \frac{g^2}{32}\,\tan^2\left(\frac{g\,x}{4}\right) + \ldots$$

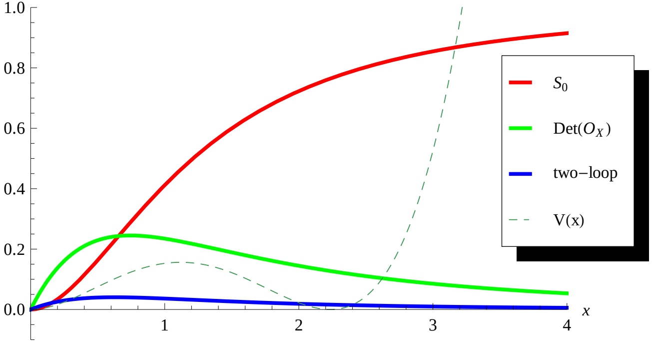
for

$$V = \frac{1}{g^2}\,(1-\cos(g\,x))$$

Anharmonic oscillator



Double well potential



Back to standard quantum mechanics

Asymptotics at large x for AHO

Schrödinger equation ($\hbar = 1, m = 1$)

$$\left(-\frac{1}{2}\partial_x^2 + V(x) - E\right)\psi \;=\; 0\;, \quad \partial_x \equiv \frac{d}{dx}$$

$$V(x) = \frac{x^2 + g^2\,x^4}{2}$$

Introducing the phase $\psi(x) = e^{-\phi(x)}$, the Riccati equation

$$\partial_x^2\phi - (\partial_x\phi)^2 \;=\; 2\,E - 2\,V(x)$$

Asymptotic expansion at $|x| \to \infty$ (A.V.T, 2010)

$$2\,\phi \;=\; \underbrace{\frac{2}{3}\sqrt{2}g|x|\,x^2 \;+\; \frac{1}{2g}x^2}_{S[X(\tau)]} - \underbrace{\log|x|^2}_{Det(O_X)} + \underbrace{\frac{1+2\,E}{\sqrt{2}\,g^2}\,\frac{1}{|x|}}_{\text{two-loop}} - \frac{1}{4\,g^3\,x^2} + \ldots$$

Identification for AHO

What is a meaning of the expansion

$$e^{-2\,\phi(x)} \;=\; \exp\left[-\left(S[X_{flucton}(\tau)]+\frac{1}{2}\mathrm{Det}(O_{X_{flucton}})-g\,B_1-\dots\right)\right]$$

In QM perturbation theory in g^2

$$\phi=\frac{x^2}{2}+g^2\;p_1(x^2)+\ldots+(g^2)^n\;p_n(x^2)+\ldots$$

where

$$p_n \;=\; a^{(n)}_{n+1}(x^2)^{n+1}+a^{(n)}_{n}(x^2)^{n}+\ldots+a^{(n)}_{k}(x^2)^{k}+\ldots+E_n x^2$$

few coeffs $a^{(n)}_{n+1}, a^{(n)}_{n}, a^{(n)}_{n-1}\ldots$ we know explicitly from 1980es

Let us make summation

$$\sum_{0}^{\infty} a_{n+1}^{(n)}(x^2)^{n+1}\,(g^2)^n \;=\; \frac{1}{3}\frac{(1+g^2\,x^2)^{\frac{3}{2}}-1}{g} \;=\; S_0[X_{flucton}(\tau)]$$

leading-log type approximation (if $x^2 \sim \log p$) !

$$\sum_{1}^{\infty} a_{n}^{(n)}(x^2)^{n}\,(g^2)^n \;=\; \log[(1+g^2\,x^2+\sqrt{1+g^2\,x^2})/2] \;=\; \frac{1}{2}\mathrm{Det}(O_{X_{flucton}})$$

next-to-leading-log type approximation, hence, we calculated determinant! And

$$\sum_{1}^{\infty} a^{(n)}_{n-1}(x^2)^{n-1}\,(g^2)^n \;=\; B_1$$

next-to-next-to-leading-log type approximation!
Needless to say we can calculate

$$\sum_{2}^{\infty} a^{(n)}_{n-2}(x^2)^{n-2}\,(g^2)^n \;=\; \tilde{B}_2$$

which is, in fact, the three-loop correction! (we failed to get directly)
Hence, the flucton loop expansion is **convergent for any real** x, g and fast-convergent !
Furthermore,

$$E \;= < e^{-\phi}|H|e^{-\phi} >$$

gives very high accuracy for any real g (no free parameters).