Uniform asymptotic evaluation of the fusion cross-section

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The Wong formula

The fusion cross section is given by,

$$\sigma_F(E) = \frac{\pi}{k^2} \sum_0^\infty (2l+1) T_l(E)$$

approximating the sum by an integral, $\sigma_F(E) = \frac{2\pi}{k^2} \int_{1/2}^\infty \lambda d\lambda T(\lambda, E)$

and using the Hill-Wheeler formula

$$T_l^{(HW)} = \frac{1}{1 + \exp\left[\frac{2\pi}{\hbar\omega}(E - V_B - \frac{\hbar^2 l(l+1)}{2\mu R^2})\right]}$$

$\lambda = l + 1/2$ is a semiclassical angular momentum

$$\sigma = \frac{\hbar\omega}{2E} R_B^2 \ln\left(1 + \exp\left[\frac{2\pi}{\hbar\omega}(E - B)\right]\right)$$

On the Wong cross section and fusion oscillations

N. Rowley[1] and K. Hagino[2,3]

Phys. Rev., C 91, 044617 (2015)

We re-examine the well-known Wong formula for heavy-ion fusion cross sections. Although this celebrated formula yields almost exact results for single-channel calculations for relatively heavy systems such as ^{16}O+^{144}Sm, it tends to overestimate the cross section for light systems such as ^{12}C+^{12}C. We generalise the formula to take account of the energy dependence of the barrier parameters and show that the energy-dependent version gives results practically indistinguishable from a full quantal calculation. We then examine the deviations arising from the discrete nature of the intervening angular momenta, whose effect can lead to an oscillatory contribution to the excitation function. We recall some compact, analytic expressions for these oscillations, and highlight the important physical parameters that give rise to them. Oscillations in symmetric systems are discussed, as are systems where the target and projectile identities can be exchanged via a strong transfer channel.

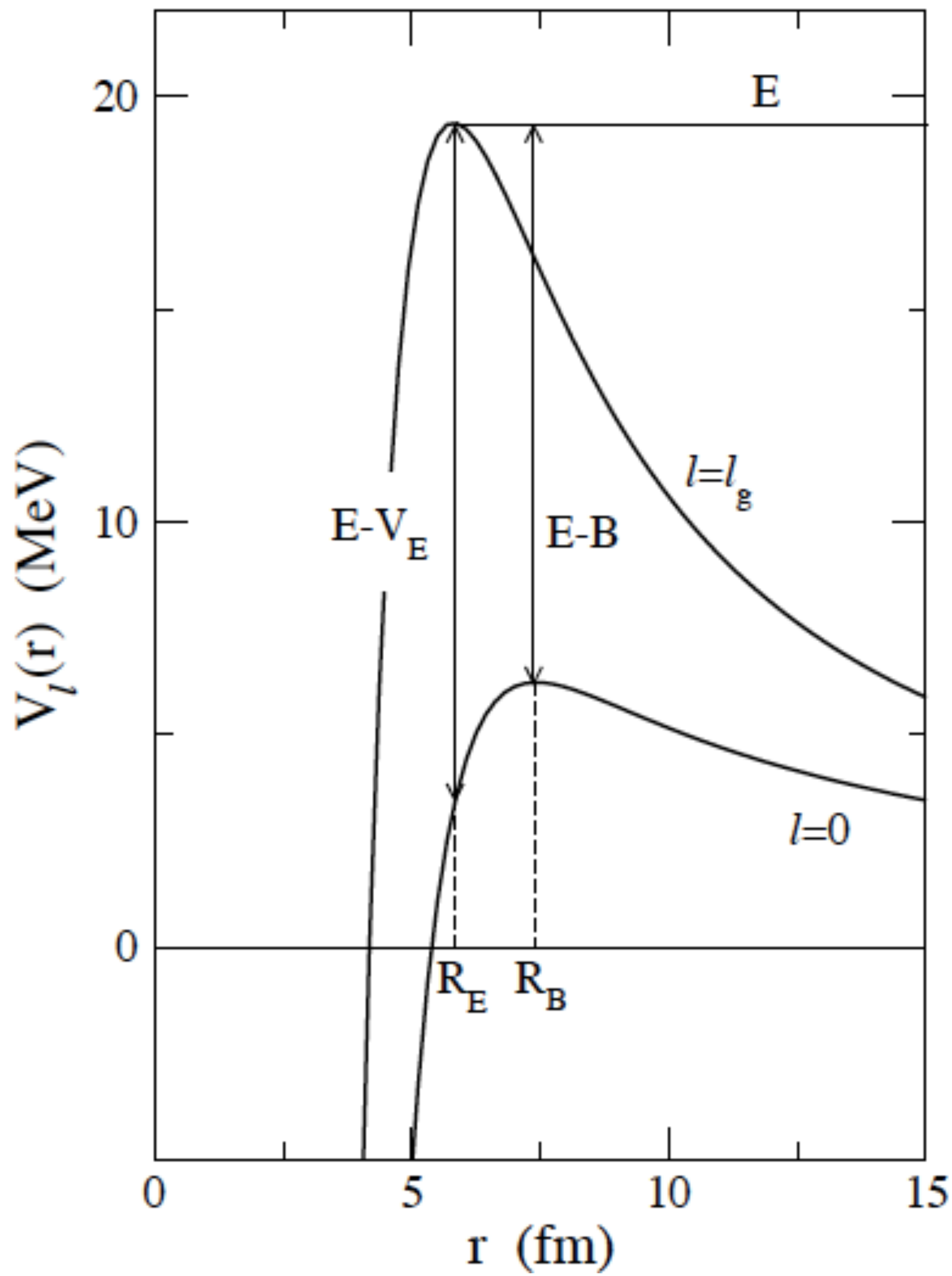
for the $l = 0$ barrier

$$\left.\frac{dV}{dr}\right|_{r=R_B} = \left.\frac{dV_N}{dr}\right|_{r=R_B} + \left.\frac{dV_C}{dr}\right|_{r=R_B} = 0$$

$$B = V(R_B) = \frac{Z_1 Z_2 e^2}{R_B}\left(1 - \frac{a}{R_B}\right)$$

$$R_B = \frac{1}{2} R_C \left(1 + \sqrt{1 - 4\frac{a}{R_C}}\right)$$

$^{12}C+^{12}C$



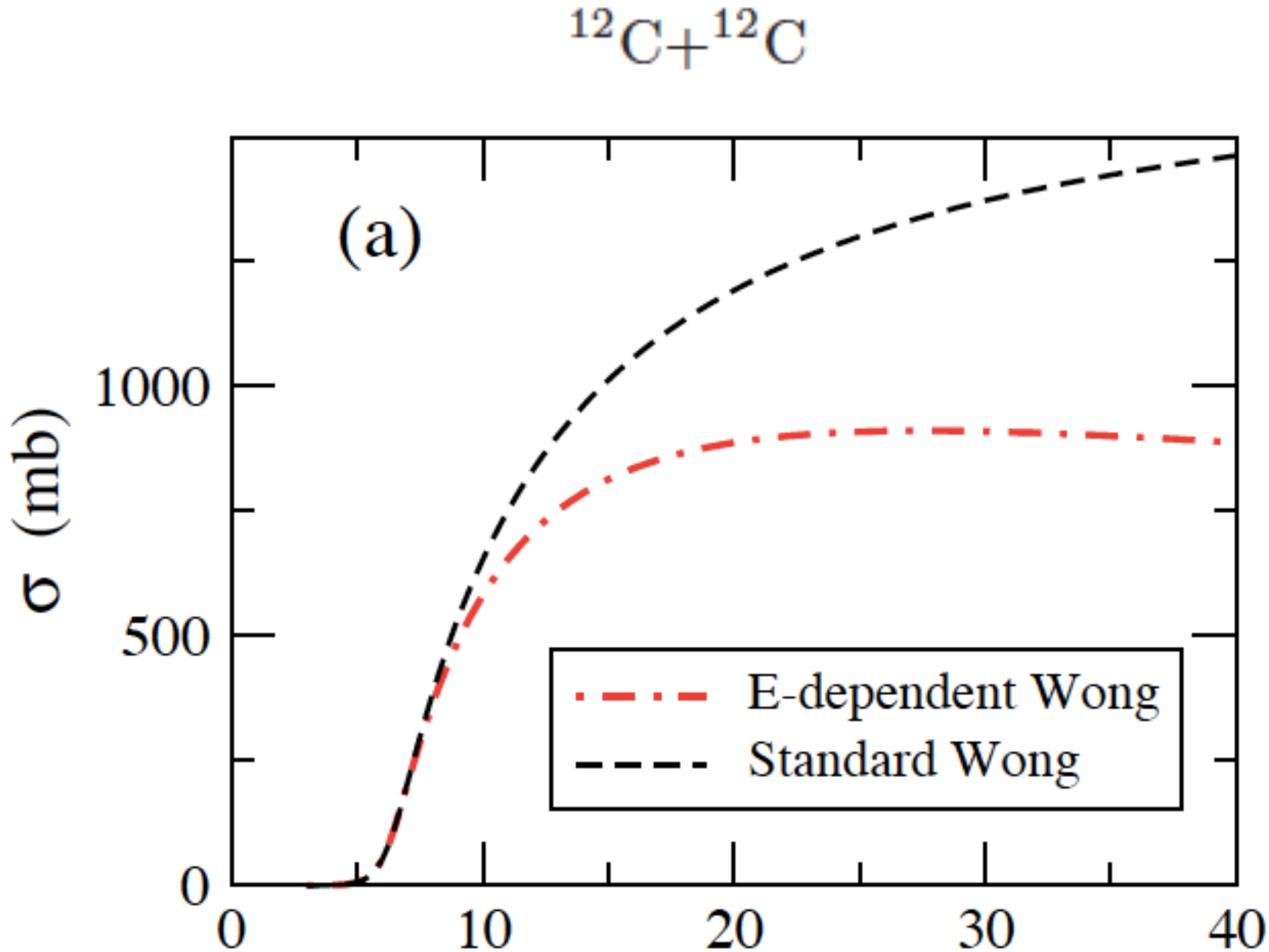
for the grazing angular momentum

$$\frac{l_g(l_g+1)\hbar^2}{2mR_E^2} = E - V_E$$

$$\sigma = \frac{\hbar\omega_E}{2E} R_E^2 \ln\left(1 + \exp\left[\frac{2\pi}{\hbar\omega_E}(E - V_E)\right]\right)$$

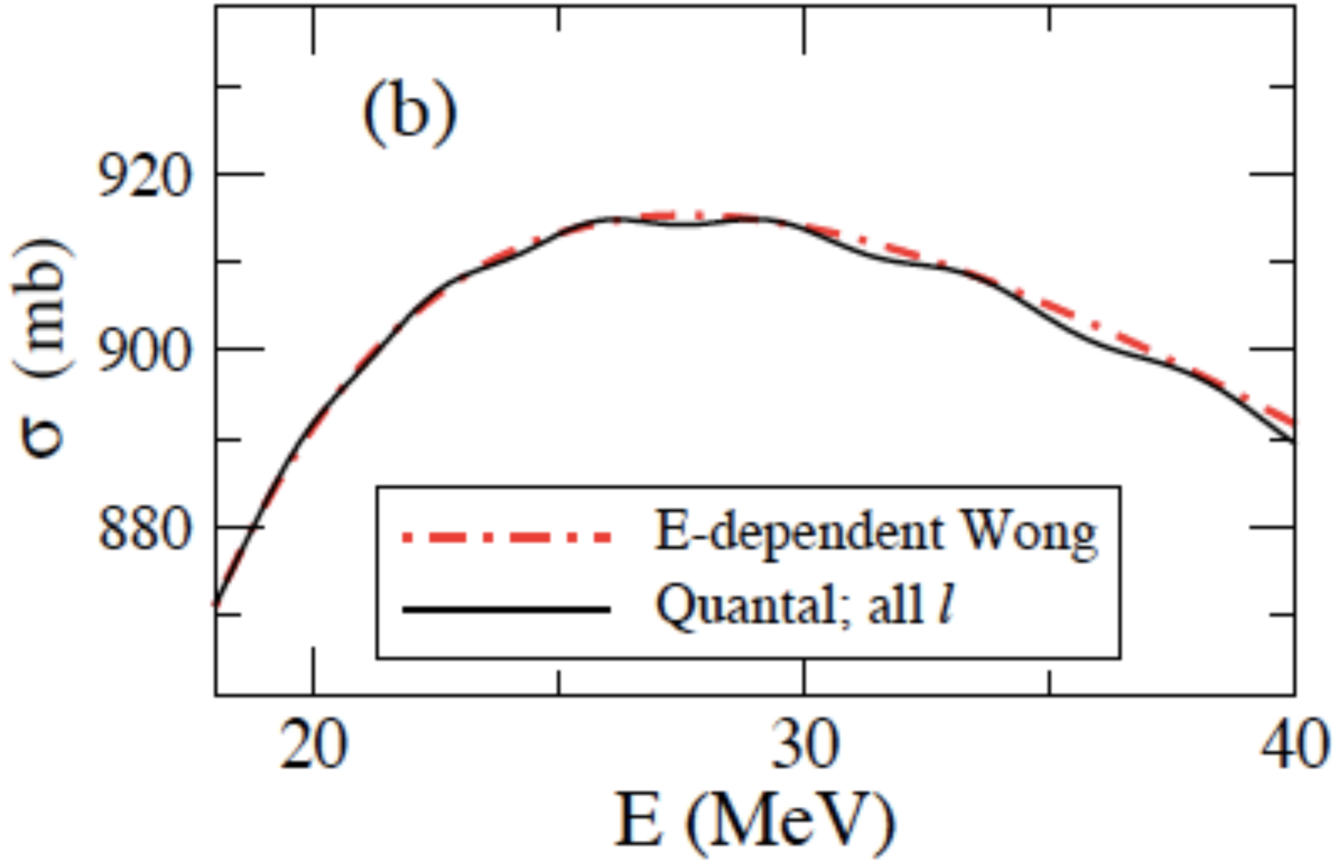
$$\hbar\omega_E = \hbar\left(-\frac{V''}{m}\right)^{1/2} = \hbar\left(-\frac{V''_C + V''_N + V''_l}{m}\right)^{1/2}$$

evaluated at $r = R_E$.



$^{12}C+^{12}C$
(a)
σ (mb)
1000
500
0
0
10
20
30
40
E-dependent Wong
Standard Wong

$^{12}C+^{12}C$



the radial Schrödinger equation for the l^th partial wave

$$\left[\frac{d^2}{dr^2} + k_l^2(r)\right] u_l(r) = 0$$

where the local wave number $k_l(r)$ given by

$$k_l(r) = \sqrt{\frac{2\mu}{\hbar^2}}\sqrt{(E - V(r) - \frac{\hbar^2 l(l+1)}{2\mu r^2}} = \sqrt{\left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right]} \equiv \sqrt{[k^2 - U_l(r)]}$$

The JWKB solution

$$u_l(r) \approx \frac{A}{\sqrt{k_l(r)}} \exp\left[-i\int_{R_0}^{r} k_l(r')dr'\right]$$

the tunneling probability obtained by Kemble

$$T_l(E) = \frac{1}{1 + e^{2\Phi^{(JWKB)}}}$$

where $$\Phi^{(JWKB)} = \int_{r_i}^{r_e} dr \kappa_l(r)$$

$$\kappa_l(r) = \sqrt{[U_l(r) - k^2]}$$

r_i and r_e are the internal and external turning points

We use the following parabolic approximation for $V_l(r)$,

$$V_l(r) \approx B_l - \frac{1}{2}\mu\omega_l^2(r - R_l)^2$$

where $B_l = V_N(R_l) + V_C(R_l) + (\hbar^2 l(l+1))/(2\mu R_l^2) \approx V_B + (\hbar^2 l(l+1))/(2\mu R_B^2)$

$$\Phi^{(HW)} = \frac{\pi}{\hbar\omega_l}\left[E - B_l\right] \approx \frac{\pi}{\hbar\omega}\left[E - V_B - \frac{\hbar^2 l(l+1)}{2\mu R^2}\right]$$

Accordingly the Hill-Wheeler expression for the tunneling probability is

$$T_l^{(HW)} = \frac{1}{1 + \exp\left[\frac{2\pi}{\hbar\omega}(E - V_B - \frac{\hbar^2 l(l+1)}{2\mu R^2})\right]}$$

$$\sigma_F(E) = \frac{2\pi}{k^2} \int_{1/2}^{\infty} \lambda d\lambda T(\lambda, E)$$

$$\sigma_F = R_B^2 \frac{\hbar\omega}{2E} \ln\left[1 + \exp\left(\frac{2\pi(E - V_B)}{\hbar\omega}\right)\right]$$

THE UNIFORM APPROXIMATION AND THE DINGLE CALCULUS

The fusion cross section is given by,

$$\sigma_F(E) = \frac{\pi}{k^2}\sum_0^{\infty}(2l+1)T_l(E)$$

using the Poisson formula,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)dy\exp\left(-2\pi iky\right)$$

$$\sigma_F = \sum_0^{\infty}\sigma_F(l) = \sum_{k=-\infty}^{\infty}(-)^k\int_0^{\infty} d\lambda\exp\left[2i\pi k\lambda\right]\sigma_F(\lambda) = \sum_{k=-\infty}^{\infty}\sigma_F(k)$$

$$\sigma_F(k=0) = \frac{2\pi}{k^2}\int_0^\infty \lambda d\lambda T(\lambda)$$

$$\sigma_F(k=\pm 1) = \frac{2\pi}{k^2}\int_0^\infty d\lambda\lambda \exp\left[\pm 2i\pi\lambda\right] T(\lambda)$$

$$\sigma_F(k=\pm 2) = \frac{2\pi}{k^2}\int_0^\infty d\lambda\lambda \exp\left[\pm 4i\pi\lambda\right] T(\lambda)$$

$$\sigma_F(k=0) = \frac{2\pi}{k^2}\int_0^\infty \frac{\lambda d\lambda}{1+\exp(2\int_{r_1}^{r_2} dr\kappa(\lambda,r))},$$

where the dependence of $\kappa(\lambda, r)$ on the angular moment is λ^2. Making and integration by parts we obtain

$$\sigma_F(k=0) = \frac{\pi}{k^2}\int_0^\infty \frac{\lambda^2 d\lambda\tau(\lambda)}{\cosh^2\left[\int_{r_1}^{r_2} dr\kappa(\lambda,r)\right]},$$

where

$$i\lambda\tau(\lambda) = i\int_{r_1}^{r_2} dr\frac{\lambda\partial\kappa(\lambda,r)}{d\lambda} = i\lambda\int_{r_1}^{r_2}\frac{dr}{r^2\sqrt{U(x,r)-k^2}}$$

is an imaginary 'angle'

Uniform Calculation

$$F(\lambda) = 2\log\left[\cosh\int_{r_1}^{r_2} dr\kappa(\lambda, r)\right] - 2\log(\lambda) = F(\lambda_s) + t^2$$

where λ_s is a stationary point given by the root of $F' = 0$

that gives the transcendental equation

$$2\tau(\lambda)\tanh\left[\int_{r_1}^{r_2} dr\kappa(\lambda, r)\right] - \frac{2}{\lambda} = 0.$$

The integral then becomes

$$\sigma_F(k=0) = \frac{\exp[-F(\lambda_s)]pi}{k^2} \int_{-\infty}^{\infty} dt C(t) \exp(-t^2),$$

where $C(t) = \tau[\lambda(t)]\frac{d\lambda}{dt}$, and the above expression is exact

Dingle

$$\int_{\text{s.p.}} e^{-F} G\, du \quad \textbf{(Stationary-point methods)}$$

$$f = (F - F_0)^{\frac{1}{2}} \rightarrow (\tfrac{1}{2} F_2)^{\frac{1}{2}} u$$

$$e^{-F_0} \int_{f=-\infty}^{\infty} e^{-f^2} G\, du$$

$$\int_{\rightarrow} e^{-F} G\, du = (2\pi/F_2)^{\frac{1}{2}} e^{-F_0} \sum_{0}^{\infty} Q_{2r}, \qquad |\text{ph}\, F_2| < \pi$$

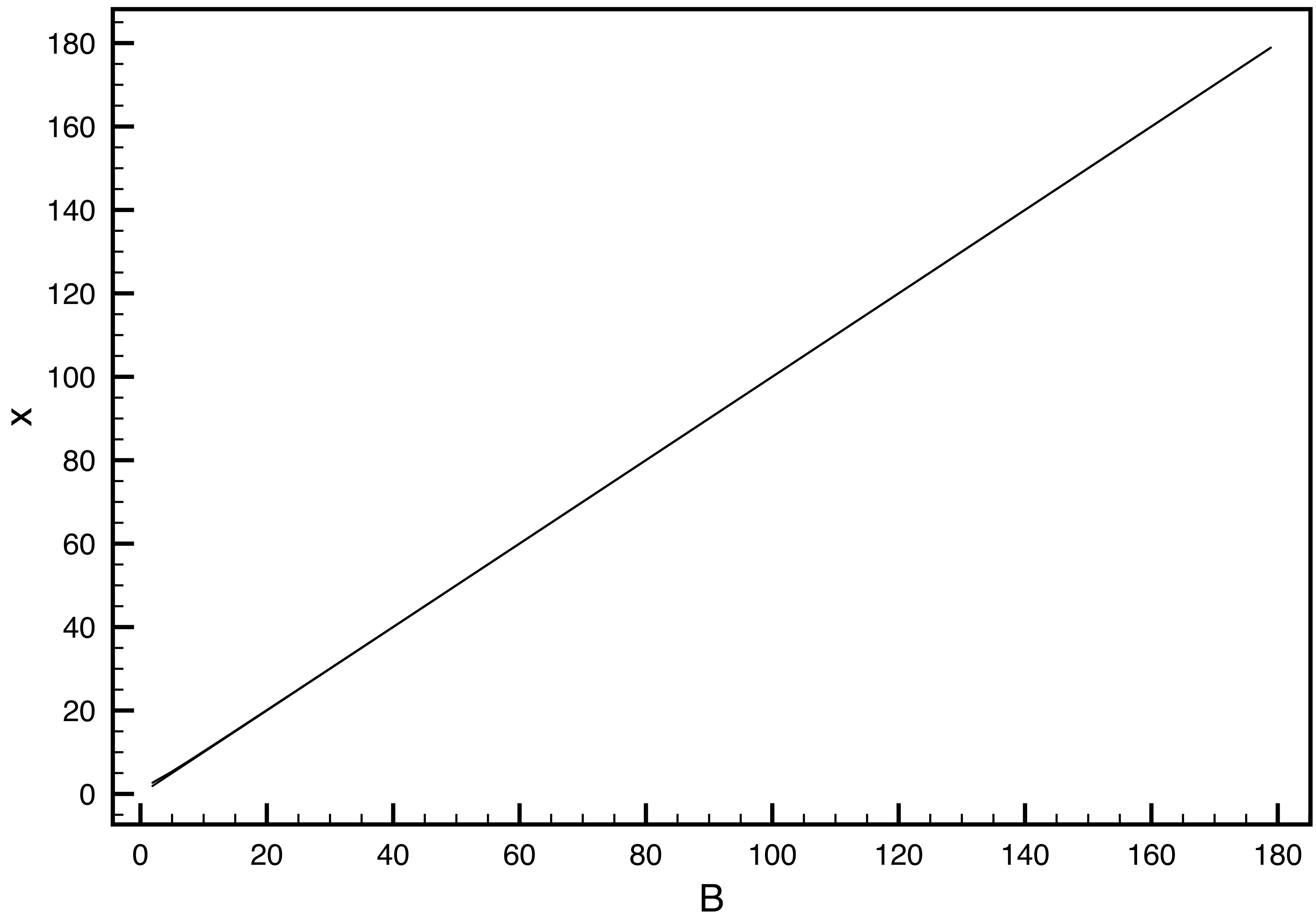
$$Q_0 = G_0, \qquad Q_1 = -\frac{\sqrt{2}}{3\sqrt{\pi}\,F_2^{\frac{3}{2}}}\{G_0F_3 - 3G_1F_2\},$$

$$Q_2 = \frac{1}{24F_2{}^3}\{G_0(5F_3{}^2 - 3F_2F_4) - 12G_1F_2F_3 + 12G_2F_2{}^2\},$$

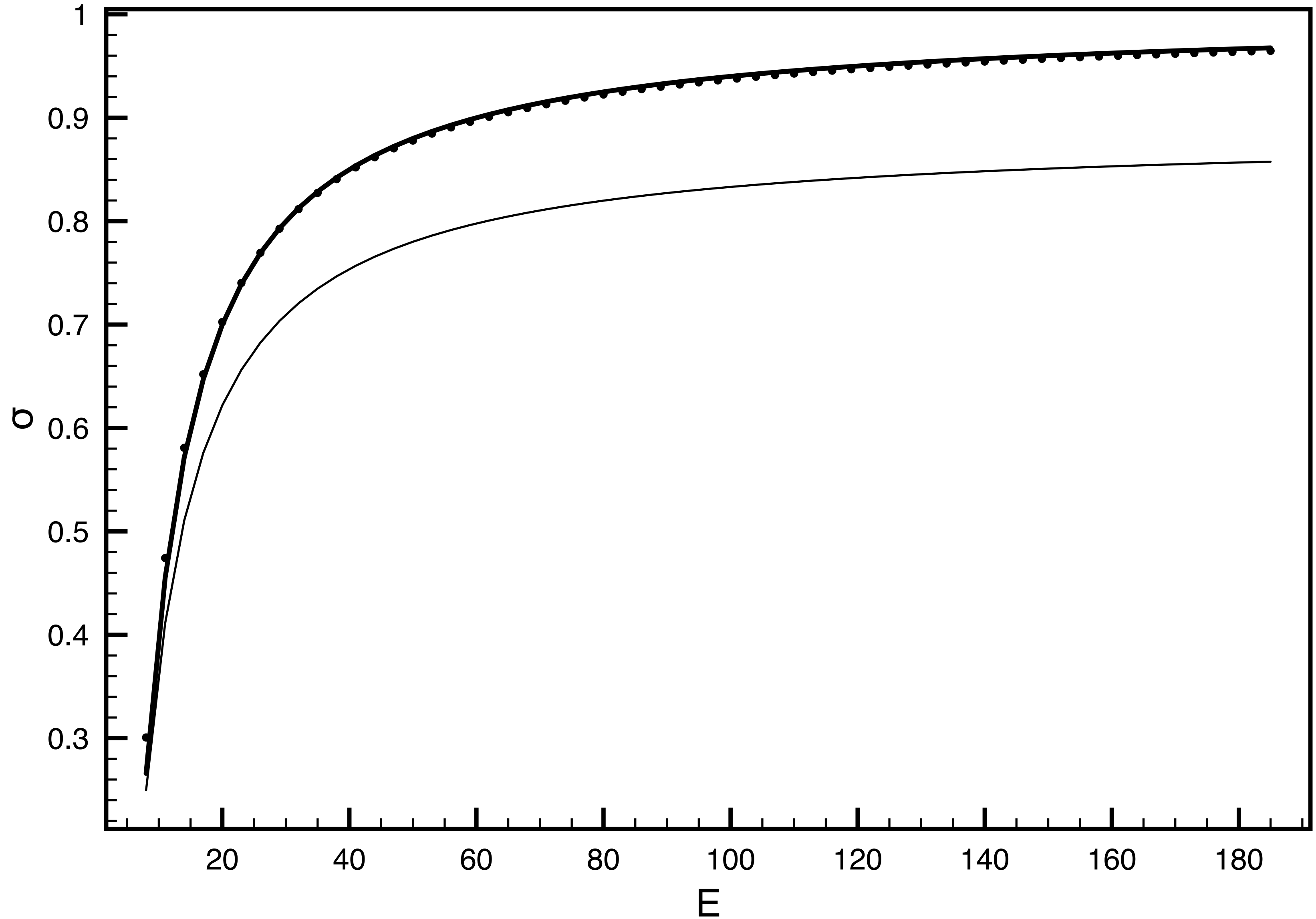
$$Q_3 = -\frac{\sqrt{2}}{135\sqrt{\pi}F_2{}^{9/2}}\{G_0(40F_3{}^3 - 45F_2F_3F_4 + 9F_2{}^2F_5) - 45G_1F_2 \times (2F_3{}^2 - F_2F_4) + 90G_2F_2{}^2F_3 - 45G_3F_2{}^3\},$$

Asymptotic evaluation of Wong formula

$$\tanh\left(\frac{x-B}{2}\right)=\frac{1}{x}$$



180
160
140
120
100
80
60
40
20
0
x
0
20
40
60
80
100
120
140
160
180
B



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
σ
20
40
60
80
100
120
140
160
180
E

Conclusion

In principle, the fusion cross-section can be calculated by using asymptotic uniform approximation to perform the integral in the semi-classical angular momentum. This will show the effective angular momentum that contributes to the fusion.