Structure of Hadrons onto the null-plane, $n \cdot x = 0$

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Outline

1. Motivations and generalities onto the null plane $n \cdot x = 0$ ($n^2 = 0$)
2. Pion and Nucleon phenomenology: Em form factors, TMD's, GPD's
3. Nakanishi perturbation-theory integral representation (PTIR) and the BS Amplitude
4. Eigenvalues, valence probability and LF distributions in ladder approximation
5. Conclusions

Motivations

- To achieve a fully covariant description of Hadrons, in Minkowski space, both in spacelike and timelike regions
- To take properly into account, as much as possible, general principles and underlying dynamics
- To make feasible numerical calculations !

The most relevant, and potentially exact, non perturbative approach:

Lattice QCD, but in Euclidean space

An integral-equation approach, like the Bethe-Salpeter Equation in Minkowski space, could represent an effective, phenomenological tool, if able to deal with refined kernel.

Helpful interplay ?

Why Phenomenology onto the null-plane, $n \cdot x = 0$

- The hyperplanes $n \cdot x = 0$ are tangent to the Light-cone: DIS dynamics automatically included.
- The most known is the plane $x^+ = t + z = 0$ ($n^\mu \equiv \{1, 0, 0, -1\}$), called Light-front (LF) plane (for $x^+ = constant$, all the points on a transverse plane are illuminated by a light pulse with the same phase)
- Poincaré covariance easily implemented within the LF framework $\rightarrow$ i) kinematical nature of LF boosts, ii) rigorous separation of internal dynamics, in analogy with the non relativistic description, iii) meaningful Fock expansion for a massive field-theoretical model (physical vacuum $\equiv$ mathematical vacuum, no zero modes).
- Light-cone wave functions: fundamental ingredients, that allow one to evaluate transition amplitudes (at a given order of approx.), taking into account the rich content of an interacting bound system.
- Within the LF framework, there exist towers of calculations: Hadron EW form factors, Compton Scattering (real and virtual), Transverse-momentum distributions, etc., but with different phenomenological approximations.

Why BSE ?

- New, effective marriage between the LF framework and the Bethe-Salpeter Equation, pointing to a covariant description of physical processes, so that one can move beyond a description based on the valence dof

$$
\begin{aligned}
|meson\rangle &= \quad |q\bar{q}\rangle + |q\bar{q}q\bar{q}\rangle + |q\bar{q}\; g\rangle..... \\
|baryon\rangle &= \underbrace{|qqq\rangle}_{\text{valence}} + \underbrace{|qqq\; q\bar{q}\rangle + |qqq\; g\rangle.....}_{\text{nonvalence}}
\end{aligned} \tag{1}
$$

- Initial step: phenomenological BS amplitudes for describing the quark-hadron vertex.
- Holy Grail: actual solutions of BSE in Minkowski space, with simple kernels that grasp the relevant features of the underlying dynamics. Key ingredient: the Nakanishi integral representation of the BS amplitude.

An example: The Mandelstam Formula for the EM current

The Mandelstam formula provides a covariant expression for the em current of Hadrons.

In the timelike region, the pion EM is given by (de Melo et al PLB. **581** (2004) 75; PRD **73**, 074013 (2006))

$$j^\mu = -\imath 2e \frac{m^2}{f_\pi^2} N_c \int \frac{d^4k}{(2\pi)^4} \Lambda_{\overline{\pi}}(k - P_\pi, P_{\overline{\pi}}) \overline{\Lambda}_\pi(k, P_\pi) \times$$
$$Tr[S(k - P_\pi)\, \gamma^5 S(k-q)\, \Gamma^\mu(k,q)\, S(k)\, \gamma^5]$$

- $S(p) = \dfrac{1}{\not p - m + \imath\epsilon}$ is the constituent quark propagator
- $\gamma_5\, \Lambda_\pi(k, P_\pi) = \lambda_\pi(k, P_\pi)$ is the pion vertex function (known caveats...), i.e. the Bethe-Salpeter Amplitude $\times\ S^{-1}(quark) S^{-1}\left(\overline{quark}\right)$
  P^μ_π and $P^\mu_{\overline{\pi}}$ are the pion momenta.
- $\Gamma^\mu(k,q)$ is the quark-photon vertex (q^μ is the virtual photon momentum)

Instead of the usual $q^+ = 0$ (Drell-Yan) frame (a popular choice within LF) for, for achieving a unified investigation of SL and TL regions we use a reference frame where

$$\boxed{q^+ > 0\ , \quad \mathbf{q}_\perp = 0}$$

(F.M. Lev, E. Pace and G. S., NPA 641 (1998) 229).

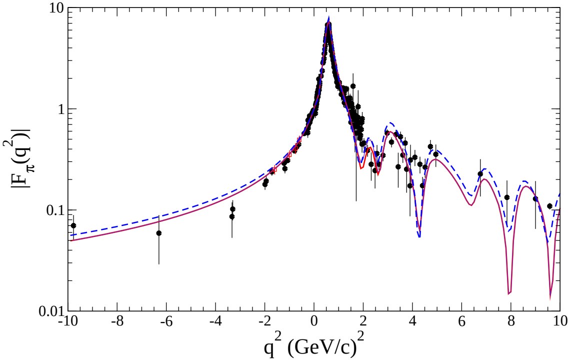
In this frame, the photon (better, its hadronic component) allows transitions from valence to nonvalence components!

In our initial investigations:

- Only the poles from the Dirac propagators are taken into account, namely i) no contributions from the analytic structure of the BS amplitude and ii) on k^--shell quark-photon vertex
- Vector Dominance Model for the pair-production term in the quark-photon vertex. Masses and valence wave functions from a realistic mass operator for vector mesons, able to reproduce masses and EM decays.
- $m_u = m_d = 0.200\ GeV$, the same values will be adopted for the nucleon EM form factors.

Pion EM Form Factor in SL and TL regions with only 2 adjusted parameters

($\Gamma^{VM}_{e^+e^-} = 150\ MeV$ for $M_{VM} >\ 2.15 GeV$ and the weight of the "instantaneous" term)



●: Compilation from R. Baldini et al. (EPJ. C11 (1999) 709, and Refs. therein.)

□: TJLAB SL data (J. Volmer et al., PRL. 86, 1713 (2001).)

Solid line: calculation with the pion w.f. from the FPZ model for the Bethe-Salpeter amplitude in the valence region ($w_{VM} = -1.0$).

Dashed line: the same as the solid line, but with the asymptotic pion w.f.

Nucleon EM Form Factors in the SL and TL regions with only 4 adjusted parameters, two parms in the BS amplitude, and two weights of the Z-diagrams, (de Melo et al PLB **671** (2009) 153).

The BS amplitude for the nucleon can be approximated as follows

$$\Phi_N^\sigma(k_1,k_2,k_3,P_N) = \imath \Big[S(k_1)\ \tau_y\ \gamma^5\ S_C(k_2)C\ \otimes\ S(k_3)\ + \\ S(k_3)\ \tau_y\ \gamma^5\ S_C(k_1)C\ \otimes\ S(k_2)\ +\ S(k_3)\ \tau_y\ \gamma^5\ S_C(k_2)C\ \otimes\ S(k_1)\Big] \\ \times\ \Lambda(k_1,k_2,k_3)\ \chi_{\tau_N}\ U_N(P_N,\sigma)$$

In the valence vertex the spectator quarks are on their-own k^--shell, and the momentum dependence, is approximated a la Brodsky (PQCD inspired), namely

$$\Psi_N(k_1,k_2,k_3) = P_N^+\ \frac{\Lambda_V(k_1,k_2,k_3)|_{on}}{[M_N^2 - M_0^2(1,2,3)]} =$$

$$= P_N^+\ \mathcal{N}\ \frac{(9\ m^2)^{7/2}}{(\xi_1\xi_2\xi_3)^p\ \ [\beta^2 + M_0^2(1,2,3)]^{7/2}}$$

The momentum dependence of the $\Lambda_V(k_1,k_2,k_3)|_{off}$ is modeled through the available LF-boost invariants.

Calculated static properties

Magnetic moments:

Proton: 2.87 (Exp. 2.793) Neutron : -1.85 (Exp. -1.913)

Proton charge radius:

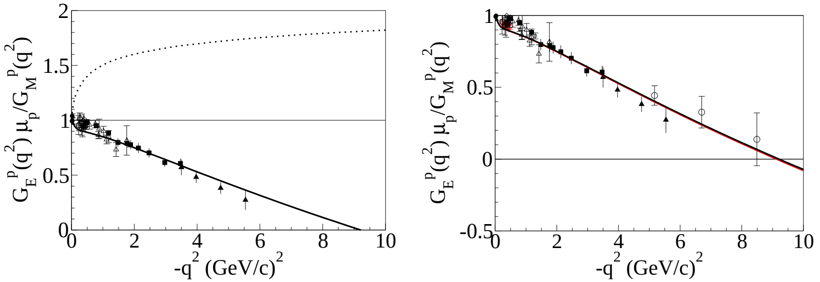
$r_p = (0.903 \pm 0.004)\ fm$ $r_p^{exp} = (0.895 \pm 0.018)\ fm$

Neutron derivative at $Q^2 = 0$

$$- \left[\frac{dG_E^n(Q^2)}{dQ^2}\right]^{th}_{Q^2=0} = (0.501 \pm 0.002)\ (GeV/c)^{-2}$$

$$- \left[\frac{dG_E^n(Q^2)}{dQ^2}\right]^{exp}_{Q^2=0} = (0.512 \pm 0.013)\ (GeV/c)^{-2}$$

Proton Ratio $\mu_p G_E^p / G_M^p$



Left Panel: Solid line: full calculation $\equiv \mathcal{F}_{val} + Z_B\ \mathcal{F}_{bare} + Z_{VM}\ \mathcal{F}_{VMD}$

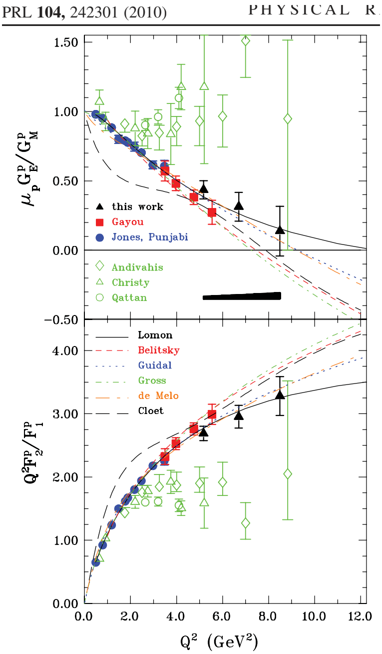
Dotted line: $\mathcal{F}_{val}$ (valence contribution only)

Data: www.jlab.org/ cseely/nucleons.html and Refs. therein.

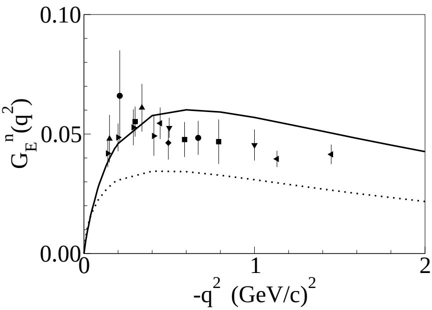
Interference between valence and pair-production contributions, i.e. higher Fock components produces our zero.

Right Panel: Solid line: only G_E^n, G_M^p and G_M^n in the fit for fixing the 4 parms.

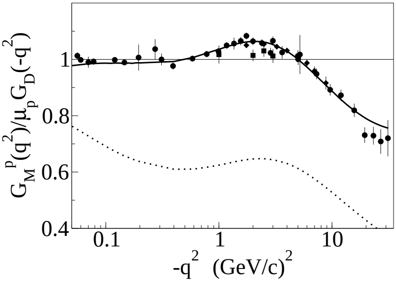
$p = 0.13$, $r = 0.17$, $Z_B = Z_{VM}^{IV} = 2.83$ and $Z_{VM}^{IS} = Z_{VM}^{IV} = 1.12$

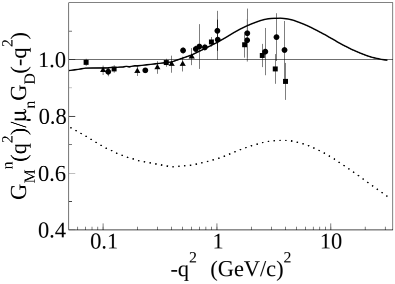

PRL 104, 242301 (2010)
PHYSICAL R
1.50
1.00
0.50
0.00
−0.50
$\mu_p G_E^p/G_M^p$
this work
Gayou
Jones, Punjabi
Andivahis
Christy
Qattan
Lomon
Belitsky
Guidal
Gross
de Melo
Cloet
4.00
3.00
2.00
1.00
0.00
$Q^2F_2^p/F_1^p$
0.0
2.0
4.0
6.0
8.0
10.0
12.0
Q^2 (GeV^2)

SL Nucleon form factors: G_E^n, G_M^p G_M^n



Solid line: full calculation $\equiv \mathcal{F}_{val} + Z_B\ \mathcal{F}_{bare} + Z_{VM}\ \mathcal{F}_{VMD}$
Dotted line: $\mathcal{F}_{val}$ (valence only)



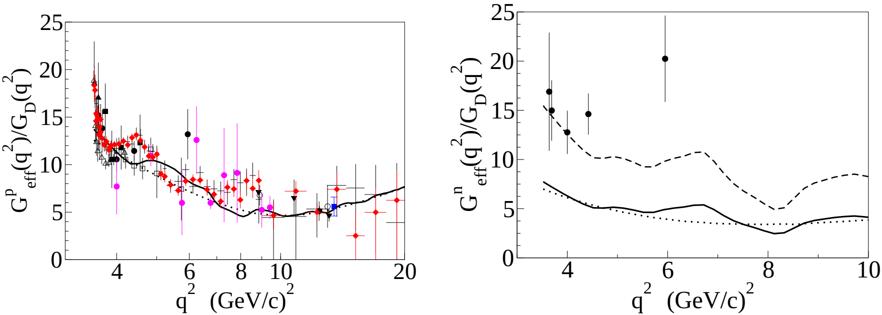


Solid line: full calculation $\equiv \mathcal{F}_{val} + Z_B\ \mathcal{F}_{bare} + Z_{VM}\ \mathcal{F}_{VMD}$

Dotted line: $\mathcal{F}_{val}$ (valence contribution only)

The pair-production contribution is essential for getting the results

$G_D = 1/[1 - q^2/(0.71\ (GeV/c)^2)]^2$



Left Panel: Solid line: full calculation - Dotted line: bare production (no VM).
Missing strength at $q^2 = 4.5\ (GeV/c)^2$ and $q^2 = 8\ (GeV/c)^2$!

Right panel: Dashed line: solid line arbitrarily $\times$ 2.

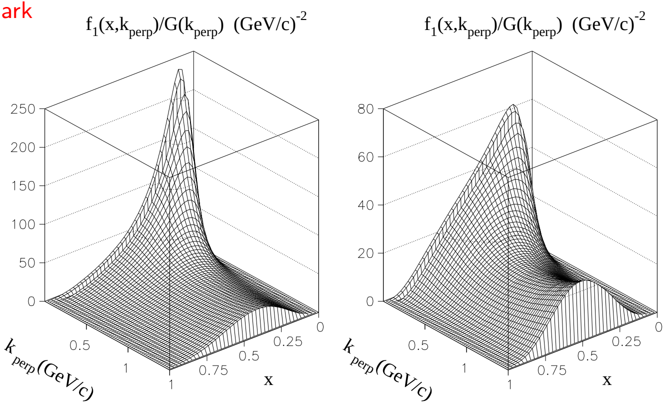
$$G_{eff}(q^2) = \sqrt{\frac{2\tau\ |G_M(q^2)|^2\ +\ |G_E(q^2)|^2}{2\tau\ +\ 1}}$$

Transverse momentum distributions in the proton: an example, the unpolarized one

$$f_1^{u(d)}(x,k_\perp) = -\frac{N_c}{(2\pi)^6 x^2}\int_0^{1-x}\frac{d\xi_2\ C_{u(d)}}{(1-x-\xi_2)\xi_2}\int\frac{d\mathbf{k}_{2\perp}}{P_N^{+2}}\ |\Psi_N(\tilde{k}_1,\tilde{k}_2,P_N)|^2\ \mathcal{H}_{u(d)}\Big|_{(k^-_{1on},k^-_{2on})}$$

$\mathcal{H}_{u(d)}$ is a proper trace of propagators $\times$ the quark current $\mathcal{I}^+_{u(d)}$

u quark / *d* quark



$G(k_{perp}) = (1\ +\ k_\perp^2/m_\rho^2)^{-5.5}$ $\qquad k_{perp} = |k_\perp|$

The decay of our $f_1(x,k_\perp)$ vs $k_\perp$ is faster than in diquark models (Jacob et al., NPA 626 (1997) 937), but slower than in factorization models (Anselmino et al., PRD 74 (2006) 074015).

Longitudinal momentum distributions

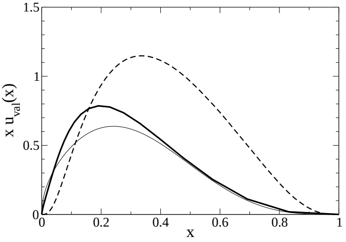
For $P'_N = P_N$ the unpolarized GPD $H_q(x, \xi, t)$ reduces to the longitudinal parton distribution function $q(x)$

$$H^q(x,0,0) = \int \frac{dz^-}{4\pi} e^{ixP_N^+ z^-} \langle P_N|\bar{\psi}_q(0)\ \gamma^+\ \psi_q(z)|P_N\rangle\Big|_{\tilde{z}=0} = q(x) = \int d\mathbf{k}_\perp\ \ f_1^q(x, k_\perp)$$

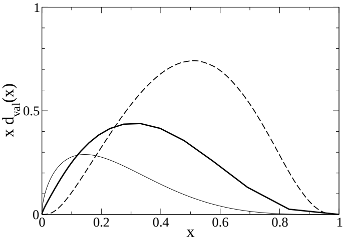
an average on the nucleon helicities is understood.

preliminary results for the Proton

u quark



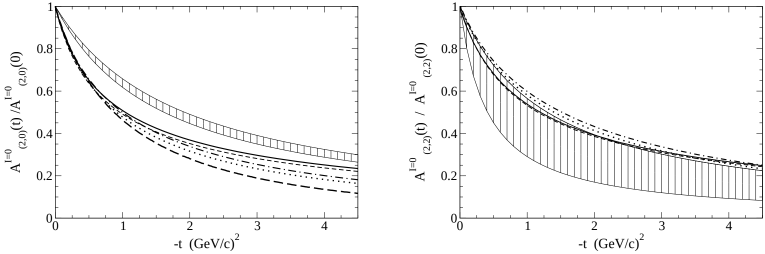
d quark



Dashed lines : our longitudinal momentum distributions
Thick solid lines : our model after evolution to $Q^2 = 1.6\ (\mathrm{GeV/c})^2$
Thin solid lines : CTEQ4 fit to data [Lai et al., PRD 51 (1995) 4753]

With the same approach, but implementing Ansatzes for the whole BS amplitude, the pion GPD's, both unpolarized and spin-dependent ones, have been evaluated.

Generalized Form Factors



Solid, dashed and dotted lines : our model results with no evolution (Frederico et al PRD **80** (2009) 054021)

Shaded area : Lattice results extrapolated at the physical pion mass and evaluated at an energy scale $\mu = 2$ GeV [Brommel et al. PRL 101 (2008) 122001].

Ratio $A(t)/A(0)$ has been reported to get rid of the multiplicative effect of evolution [Broniowsky, PRD 82 (2010) 094001]

The phenomenology we explored encourages the attempt of including more dynamics.

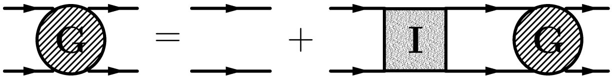
This motivates the investigation of BSE, directly in Minkowski space, as a possible tool for enriching the phenomenological description of Hadrons in non perturbative regime.

BSE in a nutshell

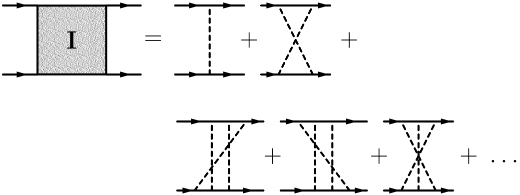
The 4-point Green's Function,

$$G(x_1, x_2; y_1, y_2) = < 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 > \quad ,$$

fulfills an integral equation $G = G_0 + G_0 \, \mathrm{I} \, G$



$\mathrm{I} \equiv$ kernel given by the infinite sum of irreducible Feynman graphs



Iterations produce all the expected contributions

Insert a complete Fock basis in

$$G(x_1, x_2; y_1, y_2) =< 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 >$$

then in the Fourier space, the bound state contribution (assuming only one non degenerate bound state for the sake of simplicity) appears as a pole, i.e.

$$G_B(k, q; p_B) \simeq \frac{i}{(2\pi)^{-4}} \; \frac{\phi(k; p_B) \; \bar{\phi}(k; p_B)}{2\omega_B(p_0 - \omega_B + i\epsilon)}$$

where $\omega_B = \sqrt{M_B^2 + |\mathbf{p}|^2}$ and $\phi(k; p_B)$ is the Bethe-Salpeter Amplitude for a bound state, in the Fourier space. In configuration space, BS Amplitude is given by

$$\langle 0| T\{\phi_1(x_1)\phi_2(x_2)\}|p_B \, \beta\rangle$$

For $p_0 \to \omega_B$ the 4-point Green's function can be approximated by

$$G \simeq G_B + \ regular \ terms$$

and one deduces from $G = G_0 \ + \ G_0 \ \mathrm{I} \ G$, the integral equation determining the BS Amplitude for a bound state, i.e. the homogeneous BS Eq.

$$\phi(k; p_B, \beta) = G_0(k; p_B, \beta) \ \int d^4q' \ \mathrm{I}(k, q'; p_B) \ \phi(q'; p_B, \beta)$$

with (nor self-energy neither vertex corrections, at the present stage)

$$G_0 = \frac{i}{(\frac{p_B}{2} + k)^2 - m^2 + i\epsilon} \ \ \frac{i}{(\frac{p_B}{2} - k)^2 - m^2 + i\epsilon}$$

Notice: $\mathrm{I}(k, q'; p_B)$, the irreducible kernel in BSE, is the same as in $G = G + 0 + G_0 \mathrm{I} G$.

Nakanishi integral representation

In the sixties, Nakanishi (PR **130**, 1230 (1963)) obtained an integral representation for N-leg transition amplitudes, based on the parametric formula for the Feynman diagrams.
Notice that a given N-leg transition amplitude is an infinite sum of Feynman graphs.
The full N-leg transition amplitude can be formally written as

$$f_N(s) = \sum_{\mathcal{G}} f_{\mathcal{G}}(s) \propto \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1-\sum_h z_h)\ \phi_N(z,\gamma)}{(\gamma - \sum_h z_h s_h)}$$

where $s \equiv \{s_h\}$ is the set of independent scalars, obtained from the external momenta, and

$$\phi_N(z,\gamma) = \sum_{\mathcal{G}} \tilde{\phi}_{\mathcal{G}}(z,\gamma)$$

is the Nakanishi weight function, that depends upon real variables!!

Within the BS framework, such an elegant representation can be exploited for obtaining

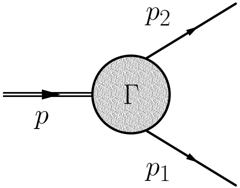
- the 3-leg transition amplitude (vertex function $\to$ bound-state BS amplitude) (Kusaka et al, PRD **56** (1997), Carbonell-Karmanov EPJA **27** (2006))
- the 4-leg one (off-shell or half-off-shell T-matrix $\to$ scattering-state BS amplitude) (FSV, PRD **85** (2012))

in simple cases

The Nakanishi integral representation of the vertex function for two scalars, interacting through the exchange of a massive scalar (massless case → Wick-Cutkosky model), reads

$$f_3(s) = \int_0^1 dz \int_0^\infty d\gamma \frac{\phi_3(z,\gamma)}{\gamma - \frac{p^2}{4} - k^2 - zk \cdot p - i\epsilon}$$

with $p = p_1 + p_2$ and $k = (p_1 - p_2)/2$. N.B. z and γ real variables.



How can the Nakanishi weight function, ϕ_3, be determined for an actual, dynamical model?

Can the Nakanishi expression, elaborated in perturbation theory , be used in a non perturbative realm, as the BS framework does (BSE is an integral equation, i.e. one has an infinite set of contributions)?

By using the Nakanishi integral representation for the BS amplitude, one can explicitly perform an analytic integration of BSE.

Adopting LF variables, $k^{\pm} = k^0 \pm k_z$ and $\mathbf{k}_\perp$, turns out to be very profitable.

After integrating both sides of the homogeneous BSE on the LF variable k^-, one gets an integral equation for the Nakanishi weight function $g_b(\gamma', z; \kappa^2)$, that, in turn, determines the BS amplitude, viz (also Carbonell-Karmanov EPJA 27, 1 (2006))

$$\int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2 - i\epsilon]^2} = \int_0^\infty d\gamma' \int_{-1}^1 dz' \ V_b^{LF}(\gamma, z; \gamma', z') g_b(\gamma', z'; \kappa^2)$$

with $V_b^{LF}(\gamma, z; \gamma', z')$ determined by the irr. kernel $\mathrm{I}(k, k', p)$!

N.B. : on the lhs, there is the valence component of the state of the interacting system (after expanding BS amplitude on the Fock basis)

Applying the uniqueness theorem for the Nakanishi weight function one gets a simpler integral equation for the weight function

$$g_b(\gamma, z; \kappa^2) = \int_0^\infty d\gamma' \int_{-1}^1 dz' \ \mathcal{V}_b(\gamma, z; \gamma', z'; \kappa^2) g_b(\gamma', z'; \kappa^2)$$

where $\mathcal{V}_b(\gamma, z; \gamma', z'; \kappa^2)$ is a new kernel, properly related to $V_b^{LF}(\gamma, z; \gamma', z')$!

$$V_b^{LF}(\gamma, z; \gamma'', z') = \int_0^\infty d\gamma' \frac{\mathcal{V}_b(\gamma', z; \gamma'', z'; \kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2 - i\epsilon]^2}$$

Numerical results for Eigenvalues and LF Distributions in ladder approx.

We have carried out a comprehensive investigation, in ladder approximation, of the simple scalar model, $\mathcal{L} = g\phi^2\chi$.

The numerical results are obtained after fixing the binding energy, $B/m = 2 - M/m$, and the mass of the exchanged scalar, μ/m, and looking for the eigenvalue (the coupling constant) and the eigenfunction (the Nakanishi weight function).

(Generalizations to cross-ladder kernels, scattering states, fermionic systems, 2+1, are in progress)

Eigenvalues

An example from Frederico et al PRD **89** (2014)

$\mu/m = 0.50$

B/m	α C-U	α LF-U	α LF-V
0.002	1.211	1.216	1.216
0.02	1.624	1.623	1.623
0.20	3.252	3.251	3.251
0.40	4.416	4.415	4.416
0.80	6.096	6.094	6.094
1.20	7.206	7.204	7.204
1.60	7.850	7.849	7.849
2.00	8.062	8.061	8.061

Values of $\alpha = g^2/(16\pi m^2)$, obtained by solving the valence-based eigenequation (LF-V) and the uniqueness-based one (LF-U). Gegenbauer $\times$ Laguerre expansion of the Nakanishi wf

C-U: from Kusaka et al PRD **56**, 5071 (1997), where uniqueness and canonical (not LF !) variables have been used and iteration method for solving the eigenequation.

Valence Probabilities

Once the Nakanishi weight functions is evaluated, one can straightforwardly obtain the BS amplitude and normalize it.

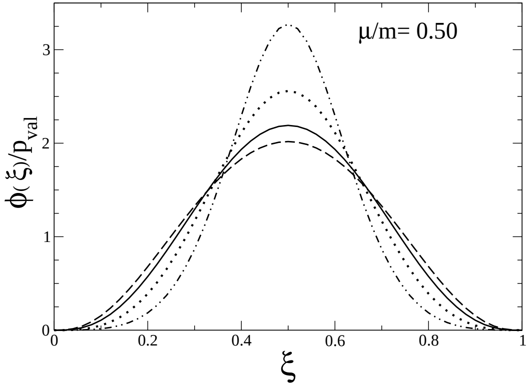
Then, the probability of the the valence wave function, $\psi_{n=2}(\xi, k_\perp)$, results properly determined and one can also calculate the LF distributions,relevant in Hadron Physics

$\mu/m = 0.50$

B/m	α	P_{val}
0.001	1.167	0.98
0.01	1.440	0.96
0.10	2.498	0.87
0.20	3.251	0.83
0.50	4.900	0.77
1.00	6.711	0.74
2.00	8.061	0.72

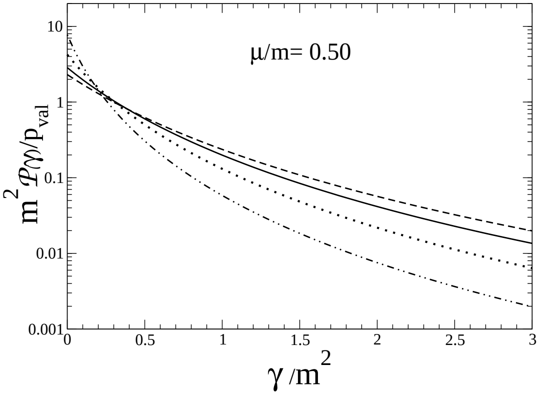
$P_{val} \rightarrow 1$ for $B \rightarrow 0$!

Longitudinal LF distributions



The longitudinal LF-distribution, $\phi(\xi) = \int dk_\perp^2 \ |\psi_{n=2}(\xi, k_\perp)|^2$, vs $\xi = k^+/M$.
Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. N.B. $\int_0^1 d\xi \ \phi(\xi) = P_{val}$

Transverse LF distributions



The transverse LF-distribution $\mathcal{P}(\gamma) = \int d\xi\ |\psi_{n=2}(\xi, k_\perp)|^2$ vs the adimensional variable γ/m^2 ($\gamma = k_\perp^2$). Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. N.B.$\int_0^\infty d\gamma\ \mathcal{P}(\gamma) = P_{val}$.

Conclusions

- The cross-fertilization between the Light-Front framework and the Nakanishi IR paves the path toward a new class of non perturbative calculations, within a rigorous field-theoretical framework (the Bethe-Salpeter Equation in Minkowski space).
- In perspective, the investigation of hadronic phenomenology could have a new tool that takes into account inputs from the underlying dynamics.
- Achieving a covariant description in Minkowski space of observables, like em form factors, GPD's and TMD's, could enrich our understanding of the corresponding calculations performed in Euclidean space.