Conformal field theories of gods and Renormalization group flows of men

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Aug. 2017
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Quotes of the day

This convoluted and baroque procedure is given the technical term renormalization. It works in practice but leaves a bitter taste in the mouth of anyone desiring simplicity of nature.

Excerpt from: C. Rovelli, *Seven Brief Lessons on Physics*

Renormalization group is a man-made thing.

Attributed to A. M. Polyakov and often heard from S. Rychkov

Outline

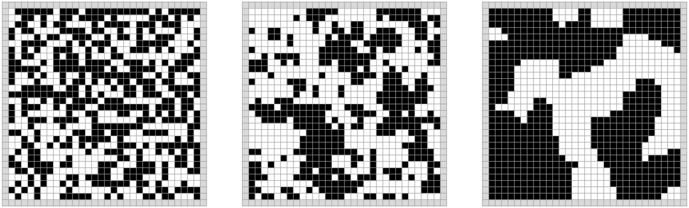
- Motivations
- Functional *perturbative* RG
- Ising universality class
- Blume-Capel universality class
- Thomas universality class

Motivation:
second order phase transitions
and critical phenomena

Example: Ising model

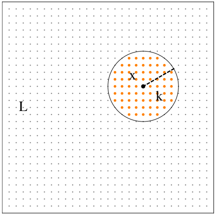
$$S = -J \sum_{\langle ij \rangle} s_i s_j + B \sum_i s_i$$

$$s_i = \pm 1$$



Effective Ginzburg-Landau theory

$$\phi(x) = \langle s_i \rangle_k$$



Mesoscopic average: $a \ll 1/k \ll L$

$$S = \int \mathrm{d}^d x \Big\{ \frac{1}{2} (\partial\phi)^2 + V(\phi) + \ldots \Big\}$$

Universality class

$$V(\phi) = \lambda\phi^4$$

- System forgets microphysics
- High predictivity: critical exponents
- Hidden symmetry: conformal

- Upper critical dimension $d_c = 4$
- Field theoretical tools: CFT and RG
- Observables: operators

$$\phi,\ \phi^2,\ \ldots,\ \partial^n\phi^m$$

CFT data

Correlators are strongly constrained. Let $O_a(x)$ be a special basis of local scalar non-derivative operators (primaries):

$$\langle O_a(x) O_b(y) \rangle = \frac{c_a \, \delta_{ab}}{|x-y|^{2\Delta_a}}$$

$$\langle O_a(x) O_b(y) O_c(z) \rangle$$
$$= \frac{C_{abc}}{|x-y|^{\Delta_a+\Delta_b-\Delta_c} |y-z|^{\Delta_b+\Delta_c-\Delta_a} |z-x|^{\Delta_c+\Delta_a-\Delta_b}}$$

4PF depends on the above!

Bootstrap program(s)

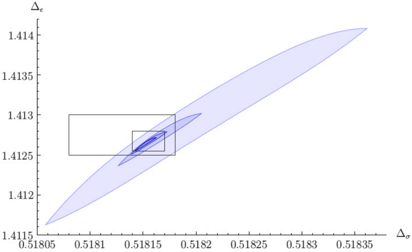
$$\sigma \sim \phi\,; \quad \epsilon \sim \frac{\phi^2}{2}$$

$$\eta = 2\Delta_\sigma - d + 2\,; \quad \nu = 1/(d - \Delta_\epsilon)$$

- Conformal bootstrap (Polyakov's) **El-Showk et al. [1203.6064]**
  - Self-consistency of CFT data using 4pf (Dolan and Osborn's)
  - Unitarity
  - Numerical and analytical

- CFT/Schwinger-Dyson bootstrap **Rychkov and Tan [1505.00963]**
  - Consistency of CFT data with free theory
  - Reconstructs 2pf and 3pf from the free theory
  - ϵ-expansion and upper critical dimensions **CSVZ [1703.04830]**

Conformal bootstrap (Polyakov's)



Simmons-Duffin [1502.02033]

- Convergence
- Clear algorithm for improvement
- Reliable error bars
- Several byproducts: OPE coefficients, structure constants
- Beats lattice error bars **Hasenbusch [1004.4486]**

Renormalization group and scheme dependence (also a motivation)

Couplings and beta functions

$$S = \sum_i \mu^{d-\Delta_i} g^i \int \mathrm{d}^d x \, \Phi_i(x) \qquad \qquad \beta^i = \mu \frac{\mathrm{d}g^i}{\mathrm{d}\mu}$$

Fixed point:

$$\beta^i(g_*) = 0$$

$$\beta^k(g_* + \delta g) = \sum_i M^k{}_i \, \delta g^i + \sum_{i,j} N^k{}_{ij} \, \delta g^i \, \delta g^j + O(\delta g^3)$$

$$M^i{}_j \equiv \left. \frac{\partial \beta^i}{\partial g^j} \right|_* \qquad N^i{}_{jk} \equiv \frac{1}{2} \left. \frac{\partial^2 \beta^i}{\partial g^j \partial g^k} \right|_*$$

Scaling operators at the FP

Scaling analysis:

$$\sum_{i,j} \mathcal{S}^a{}_i \, M^i{}_j \, (\mathcal{S}^{-1})^j{}_b = -\theta_a \delta^a{}_b \qquad \tilde{C}^a{}_{bc} = \sum_{i,j,k} \mathcal{S}^a{}_i \, N^i{}_{jk} \, (\mathcal{S}^{-1})^j{}_b \, (\mathcal{S}^{-1})^k{}_c$$

Critical exponents:

$$\theta_i = d - D_i = d - \delta_i - \tilde{\gamma}_i \qquad D_i \simeq \Delta_i$$

Scaling form:

$$S = S_* + \sum_a \mu^{\theta_a} \lambda^a \int \mathrm{d}^d x \, \mathcal{O}_a(x) + O(\lambda^2) \qquad \mathcal{O}_a = \sum_i (\mathcal{S}^{-1})^i{}_a \, \Phi_i$$

Conformal perturbation theory

Our perturbative expansion

$$\beta^a = -(d - \Delta_a)\lambda^a + \sum_{b,c} \tilde{C}^a{}_{bc}\, \lambda^b \lambda^c + O(\lambda^3)$$

Compare with Cardy's scheme*

$$\beta^a = -(d - \Delta_a)\lambda^a + \sum_{b,c} C^a{}_{bc}\, \lambda^b \lambda^c + O(\lambda^3)$$

where $C^a{}_{bc}$ come from the OPE

$$\mathcal{O}_a(x)\mathcal{O}_b(y) = \sum_c \frac{1}{|x-y|^{\Delta_a + \Delta_b - \Delta_c}} C^c{}_{ab} \mathcal{O}_c(x)$$

Are $\tilde{C}^a{}_{bc} = C^a{}_{bc}$? Well...

Transformation properties (a way around the caveats)

$$\bar{g}^i = \bar{g}^i(g)$$

$$\bar{M}^i{}_j = \frac{\partial \bar{g}^i}{\partial g^l}\, M^l{}_k\, \frac{\partial g^k}{\partial \bar{g}^j} \qquad \Longrightarrow \qquad \bar{\theta}_a = \theta_a \quad \text{and} \quad \bar{\Delta}_a = \Delta_a$$

$$\bar{\tilde{C}}^c{}_{ab} = \tilde{C}^c{}_{ab} + \frac{1}{2}\left(\theta_c - \theta_a - \theta_b\right) \frac{\partial^2 g^c}{\partial \bar{g}^l \partial \bar{g}^m} \frac{\partial \bar{g}^l}{\partial g^a} \frac{\partial \bar{g}^m}{\partial g^b}$$

$$\bar{\tilde{C}}^c{}_{ab} = \tilde{C}^c{}_{ab} \qquad \text{if} \quad \text{either} \qquad \theta_c - \theta_a - \theta_b = 0$$

$$\text{or} \qquad \frac{\partial^2 g_c}{\partial g'_p \partial g'_q} = 0$$

Functional perturbative RG
and the Ising universality class

Motivations

- $\theta_c - \theta_a - \theta_b = 0$ unlikely to happen with any generality
- Arbitrary scheme $\frac{\partial^2 g_c}{\partial g'_p \partial g'_q} \neq 0$

Codello et al. [1310.7625]

We seek for:

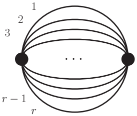
- Universality
- Simplicity

CSVZ [1705.05558]

Path integral

$$S[\phi] = \int \mathrm{d}^d x \Big\{ \frac{1}{2}(\partial\phi)^2 + V(\phi) \Big\}$$

$$W = \int \mathrm{D}\chi \, \mathrm{e}^{-S[\phi+\chi]} = \int \mathrm{D}\chi \, \mathrm{e}^{-S_0[\phi+\chi] - \int V(\phi+\chi) - J\cdot\chi}$$

$W^{(2)} = \sum_r \frac{1}{r!}$ 

in $d = 4 - \epsilon$

$$S_{\text{c.t.}} = \int \mathrm{d}^d x \Big\{ \frac{1}{2(4\pi)^2\epsilon} V^{(2)}(\phi)^2 - \frac{1}{12(4\pi)^4\epsilon} V^{(4)}(\phi)^2(\partial\phi)^2 \Big\}$$

Functional perturbative RG

At LO

$$\beta_V = \frac{1}{2}\frac{(V^{(2)})^2}{(4\pi)^2} \qquad\qquad \beta_Z = -\frac{1}{6}\frac{(V^{(4)})^2}{(4\pi)^4}$$

Define dimless $v(\varphi) = (4\pi)^2\, k^{-4} V(\varphi k^{(d-2)/2} Z_0)$

$$\beta_v = -\,4v + \varphi v^{(1)} + \epsilon\left(v - \frac{1}{2}\varphi v^{(1)}\right) + \frac{1}{2}\eta\varphi v^{(1)}$$

$$z(0) = 1 \qquad \Longrightarrow \qquad \eta = -\mu\partial_\mu \log Z_0 = \frac{1}{6}(v^{(4)}(0))^2$$

β_V as the generator of gamma-functions in $d = 4$

Scaling limit at GFP as a function of the critical coupling g:

$$v(\varphi) = g\varphi^4 + \sum_{k\neq 4} g_k \varphi^k$$

$$\beta_v \simeq \beta_g \varphi^4 - \sum_{k\neq 4} \left(d - \delta_k - \tilde{\gamma}_k(g)\right) g_k \varphi^k + \sum_{k,r,s\neq 4} \tilde{C}^k{}_{rs}(g) g_r g_s \varphi^k + \dots$$

Now assemble GFP data to describe non-trivial FP in $d < 4$.

Ising critical exponents in $d = 4 - \epsilon$

Very democratically:

$$v(\varphi) = \sum_{k\geq 0} g_k \varphi^k$$

$$\beta_v = 0 \qquad \Longrightarrow \qquad v(\varphi) = g\varphi^4 \qquad \text{with} \qquad g = \frac{\epsilon}{72} + \frac{17\epsilon^2}{1944}$$

$$\theta_i = 4 - i - \frac{1}{6}(i^2 - 4i + 6)\epsilon + \frac{1}{324} i(18i^2 - 70i + 49)\epsilon^2 - \frac{2}{27}\epsilon^2 \delta_{i,4}$$

$$\eta = 2 - d + 2\theta_1 \qquad \nu = (\theta_2)^{-1} \qquad \omega = -\theta_4 = \beta_g'$$

Some of Ising's CFT data

$$\Delta_i \simeq d - \theta_i$$

in CFT notation

$$\Delta_1 = \Delta_\sigma \qquad\qquad \Delta_2 = \Delta_\epsilon$$

${C^i}_{jk}$ are estimated whenever $i + j - k = 4$
(that is $\theta_k - \theta_i - \theta_j = 0$ at Gaussian point)

$$\tilde{C}^1{}_{23} = 6 - 2\epsilon \qquad \tilde{C}^1{}_{14} = \frac{2}{3}\epsilon \qquad \tilde{C}^2{}_{33} = 18 - 15\epsilon$$

Agree with both CFT and other QFT estimates.
There is much more to see. **CSVZ [1705.05558]**

Intermezzo:
Relation with non-perturbative FRG

Wetterich equation in the LPA

$$k\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\Gamma^{(2)} + R_k\right) k\partial_k R_k$$

Choosing a specific cutoff:

$$\beta_V = k\partial_k V = \frac{2}{(4\pi)^{d/2} d\Gamma(\frac{d}{2})} \frac{k^{d+2}}{k^2 + V''}$$

$$\beta_V = \frac{2}{(4\pi)^{d/2} d\Gamma(\frac{d}{2})} \left\{ k^d - k^{d-2}V'' + k^{d-4}(V'')^2 - k^{d-6}(V'')^3 + \dots \right\}$$

Logarithmic singularities

In $d = 4$

$$\Delta V = -\int^{\Lambda} \frac{\mathrm{d}k}{k}\beta_V \sim -\int^{\Lambda} \frac{\mathrm{d}k}{k} k^{d-4}(V'')^2 \sim -(V'')^2 \log \Lambda$$

$$\beta_V = \frac{2}{(4\pi)^{d/2} d\Gamma(\frac{d}{2})}\Big\{k^d - k^{d-2}V'' + k^{d-4}(V'')^2 - k^{d-6}(V'')^3 + \dots\Big\}$$

$$\supset \frac{1}{2(4\pi)^2}(V'')^2 \qquad \text{in} \quad d = 4$$

Chop all other terms ($\overline{\text{MS}}$)

$$\beta_V = \frac{1}{2(4\pi)^2}(V'')^2 \qquad \text{Ising}$$

Universal terms

Extract the perturbative flow in various dimensions:

$$\begin{array}{lllll} \beta_V = & -\frac{1}{4\pi} V'' & \text{in} & d=2 & \mathrm{Sine-Gordon} \\ \beta_V = & \frac{1}{2(4\pi)^2}(V'')^2 & \text{in} & d=4 & \mathrm{Ising} \\ \beta_V = & -\frac{1}{6(4\pi)^3}(V'')^3 & \text{in} & d=6 & \mathrm{Lee-Yang} \end{array}$$

The generator

$$\beta_V = \quad k^d \, \mathrm{e}^{-\frac{V''}{4\pi k^2}}$$

The propertime flow equation was one of the first and most successful tools of functional renormalization, but its appearance in this context shows beyond any reasonable doubt that *it is the most fundamenta among all of functional renormalization approaches.*

A less known critical model: the Blume-Capel universality class

Families of multicritical models

$\mathbb{Z}_2$ even unitary models ϕ^{2n}

$$d_{2n} = \frac{2n}{n-1} = \infty\,,\, 4\,,\, 3\,,\, \frac{8}{3}\,,\, \ldots\,,\, 2$$

$\mathbb{Z}_2$ odd non-unitary models ϕ^{2n+1}

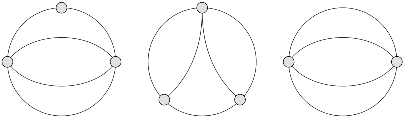
$$d_{2n+1} = 2 + \frac{4}{2n-1} = 6\,,\, \frac{10}{3}\,,\, \frac{14}{5}\,,\, \ldots\,,\, 2$$

Gracey [1703.09685]

One of them caught our eyes.

The quintic model

$d_5 = \frac{10}{3} > 3$: non-trivial in $d = 3$!? **Zambelli et. al. [1612.087397]**



Blume-Capel universality class **CSVZ [1706.06887]**

$$\beta_v = -\frac{10}{3}v + \frac{2}{3}\varphi v' + \epsilon\left(v - \frac{1}{2}\varphi v'\right) + \frac{\eta}{2}\varphi v'$$
$$+ \frac{1}{3}v^{(2)}(v^{(4)})^2 - \frac{3}{2}(v^{(3)})^2 v^{(4)}$$

This is a tricritical generalization of Lee-Yang universality class...

RG & CFT data

Spectrum:

$$\theta_i = \frac{10}{3} - \frac{2i}{3} + \epsilon\Big(-1 + \frac{i}{2}\Big) - \tilde{\gamma}_i$$
$$\tilde{\gamma}_i = \frac{\epsilon}{153}\Big(\frac{52}{5}i - \frac{139}{12}i^2 - \frac{1}{2}i^3 + \frac{19}{12}i^4 - \delta_{i,5}\Big)$$

Setting $\epsilon = 1/3$ we find

$$\eta = -4.357 \cdot 10^{-4} \qquad \sigma = \theta_4/\theta_1 = 0.2030$$
$$\nu = (\theta_2)^{-1} = 0.4977 \qquad \zeta = \theta_3/\theta_1 = 0.5596$$

Deviations from MF $\sim 10^{-4}$ are accessible (maybe),
but to trigger attention we needed a lattice model as bait...

Why Blume-Capel universality?

Blume-Capel spin-1 chain:

$$\mathcal{H} = \sum_j \left\{ -S_j^z S_{j+1}^z + \alpha (S_j^z)^2 + \beta S_j^x + \mathrm{i} h S_j^z \right\}$$

If promoted to a grid should have enough dof for the class.

In fact, non-Hermitian tricriticality at imaginary magnetic field in $d = 1 + 1$ **von Gehlen [hep-th/9402143]**

Emergent Supersymmetry

Gross-Neveu-Yukawa model

Let ψ be a Majorana $3d$ fermion:

$$S^{\mathrm{Y}}[\phi,\psi,\bar{\psi}] = \int \mathrm{d}^3x \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi + U(\phi) + \frac{1}{2}H(\phi)\bar{\psi}\psi \right\}$$

Discrete chirality:

$$\phi \to -\phi\,, \qquad \psi \to -\psi\,, \qquad \bar{\psi} \to \bar{\psi}$$

- Conjectured to have emergent SUSY in the IR if chiral **Thomas**
- Tested at $2-3$ loops **[1607.05316], [1703.08801]**
- Potential experimental realizations **Grover et al. [1301.7449]**
- Understanding is still unsatisfactory

Reparametrization of the GNY models

$$V(\phi) = \frac{1}{2}W'(\phi)^2\,, \qquad H(\phi) = W''(\phi) + Y(\phi)$$

$$S^{\mathrm{Y}}[\phi,\psi,\bar{\psi}] = \int \mathrm{d}^3x \Big\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}W'(\phi)^2 + \frac{1}{2}W''(\phi)\bar{\psi}\psi + \frac{1}{2}Y(\phi)\bar{\psi}\psi \Big\}$$

$$\mathrm{e}^{iS^{\mathrm{Y}}[\phi,\psi,\bar{\psi}]} = \int DF\,\mathrm{e}^{iS[\phi,\psi,\bar{\psi},F]}$$

Include an auxiliary (quadratic) field F with e.o.m. $F = W'(\phi)$

GHWZ [1705.08312]

Thomas model

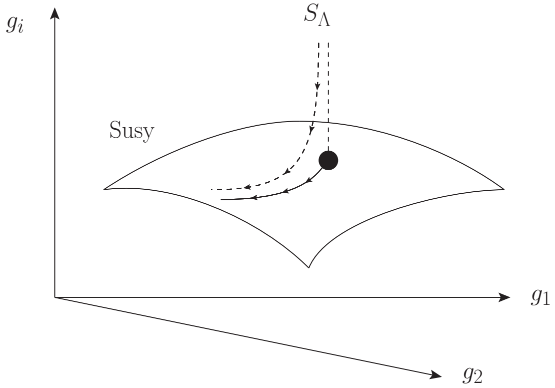
$$S[\phi,\psi,\bar{\psi},F] = \int \mathrm{d}^3x \Big\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}F^2 + F\,W'(\phi) + \frac{1}{2}W''(\phi)\bar{\psi}\psi + \frac{1}{2}Y(\phi)\bar{\psi}\psi \Big\}$$

$\mathcal{N} = 1$ SUSY transformation:

$$\delta\phi = \bar{\theta}\psi\,, \qquad \delta\psi = (F + i\partial\!\!\!/\phi)\,\theta\,, \qquad \delta F = i\bar{\theta}\partial\!\!\!/\psi$$

SUSY is broken by $Y(\phi)$

SUSY as an emergent symmetry



- Does the critical point lie on the SUSY surface?
- Are the deformations from SUSY relevant or irrelevant?

RG steps and broken off-shell SUSY

RG step on the SUSY surface:

$$S^{\mathcal{N}=1} \to S^{\mathcal{N}=1} - \frac{\delta k}{k} \int \mathrm{d}^d x \Big\{ \beta'_W F + \frac{1}{2} \beta''_W \bar{\psi} \psi \Big\}$$

Broken SUSY RG step:

$$S \to S - \frac{\delta k}{k} \int \mathrm{d}^d x \Big\{ A(\phi) F + \frac{1}{2} B(\phi) \bar{\psi} \psi + C(\phi) \Big\}$$

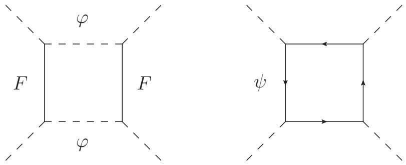
An F field redefinition can cancel $C(\phi)$

$$\begin{aligned} S &\to S - \frac{\delta k}{k} \int \mathrm{d}^d x \Big\{ A(\phi) F + \frac{1}{2} B(\phi) \bar{\psi} \psi + C(\phi) - \frac{\delta S}{\delta F} \beta_F \Big\} \\ F &\to F - \beta_F \frac{\delta k}{k} \end{aligned}$$

Idea comes from bosonization of four-Fermi interactions.

Gies and Wetterich [hep-th/0107221]

RG steps with the F field redefinition

$C(\phi) \propto$  $\propto \quad (w'')^4 - h^4$

$$S \to S - \frac{\delta k}{k} \int \mathrm{d}^d x \Big\{ \Big[A(\phi) - \frac{C(\phi)}{W'(\phi)} \Big] F + \frac{1}{2} B(\phi) \bar{\psi} \psi \Big\}$$

Redefinition ensures that $\langle F \rangle = \langle W'(\varphi_0) \rangle = 0$, but not 1PI

Pawlowski

Does it make sense? Looking for a *pragmatic* answer.

RG flow

$$\begin{aligned}
\beta_v &= -4v + \phi v' + \epsilon\Big(v - \frac{1}{2}\phi v'\Big) + \frac{\eta_\phi}{2}\phi v' \\
&\quad + \frac{1}{2(4\pi)^2}(v'')^2 - \frac{1}{2(4\pi)^2}h^4 \\
\beta_h &= -h + \phi h' - \epsilon\frac{1}{2}\phi h' + \eta_\psi h + \frac{\eta_\phi}{2}\phi h' + \frac{2}{(4\pi)^2}h(h')^2
\end{aligned}$$

$$\begin{aligned}
\beta_w &= -3w + \phi w' + \epsilon\Big(w - \frac{1}{2}\phi w'\Big) + \frac{\eta_\phi}{2}\phi w' + \frac{1}{3(4\pi)^2}(w'')^3 \\
&\quad + \int_0^\phi \mathrm{d}x\, \frac{w''(x)^4 - h(x)^4}{2(4\pi)^2 w'(x)} \\
\beta_h &= -h + \phi h' - \frac{\epsilon}{2}\phi h' + \eta_\psi h + \frac{\eta_\phi}{2}\phi h' + \frac{2}{(4\pi)^2}h(h')^2
\end{aligned}$$

Equivalence and properties

Up to irrelevant operators:

$$w'\beta_{w'} \simeq \beta_v$$

One has no chance to prove this directly in $d = 3$.
Take it as a guiding principle instead!

SUSY implies more predictive power: ν and η are related:

$$\frac{1}{\nu} = \frac{d - \eta}{2}$$

Numerical estimates using FRG

Confidence gained from perturbative setup can be applied to nonperturbative FRG in $d = 3$.

	FRG $n=1$	FRG $n=2$	ϵ^2	ϵ^3	FRG Yukawa	CB
η	0.174	0.167	0.184	0.162	0.185*	0.164
ν^{-1}	1.385	1.395	1.408	1.419	1.29	1.418*
ω	0.765	0.782	0.700*	0.885*	0.796	–
$\tilde{\omega}$	0.809	0.831	0.909*	1.407*	1.09	–

Conclusions

Functional perturbative RG

- allows to setup the computation of CFT data
  (with some limitations),
- is a tool to explore new universality classes,
- and gives a framework to explore some new technique